Logarithmic Superconformal Minimal Models

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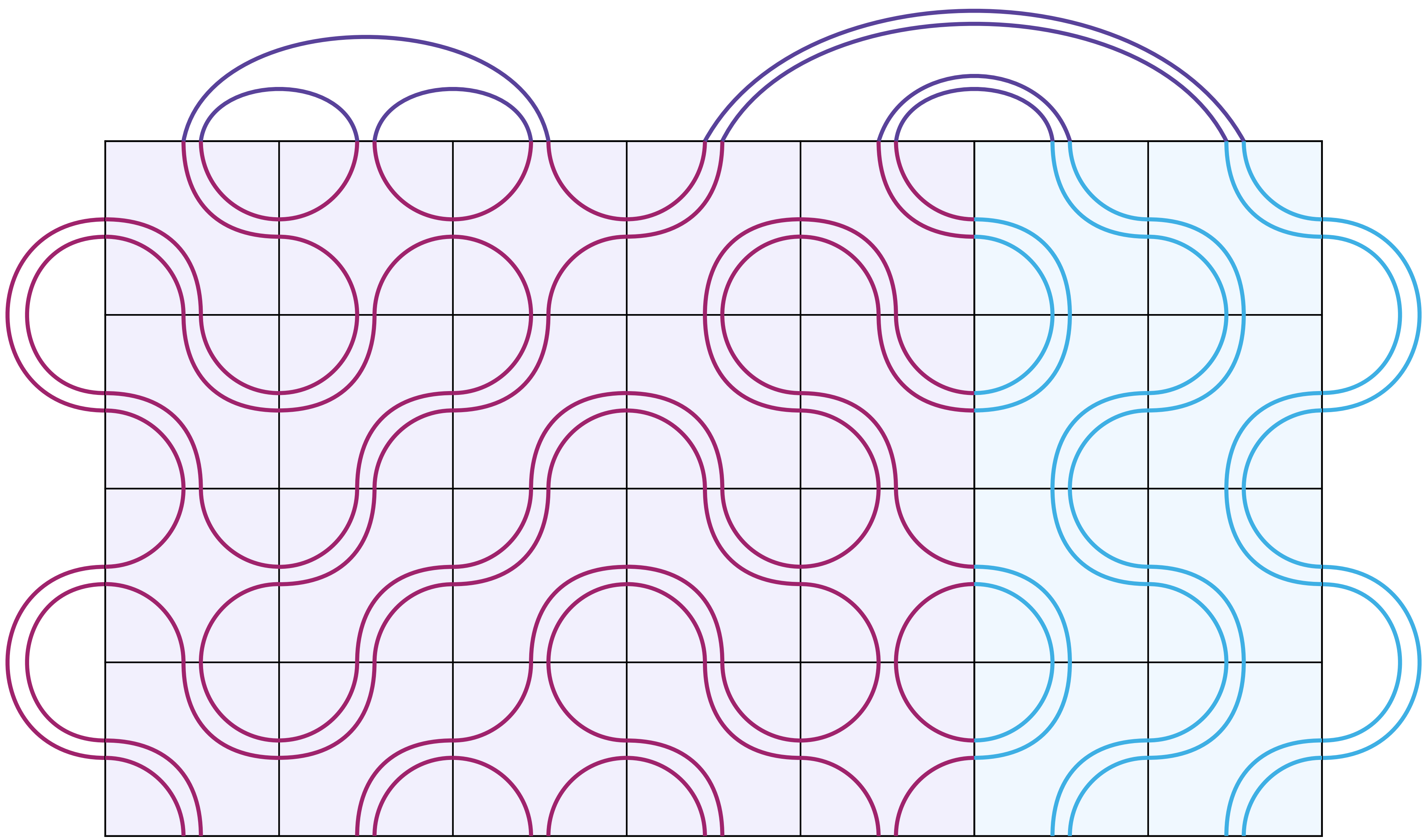


Outline

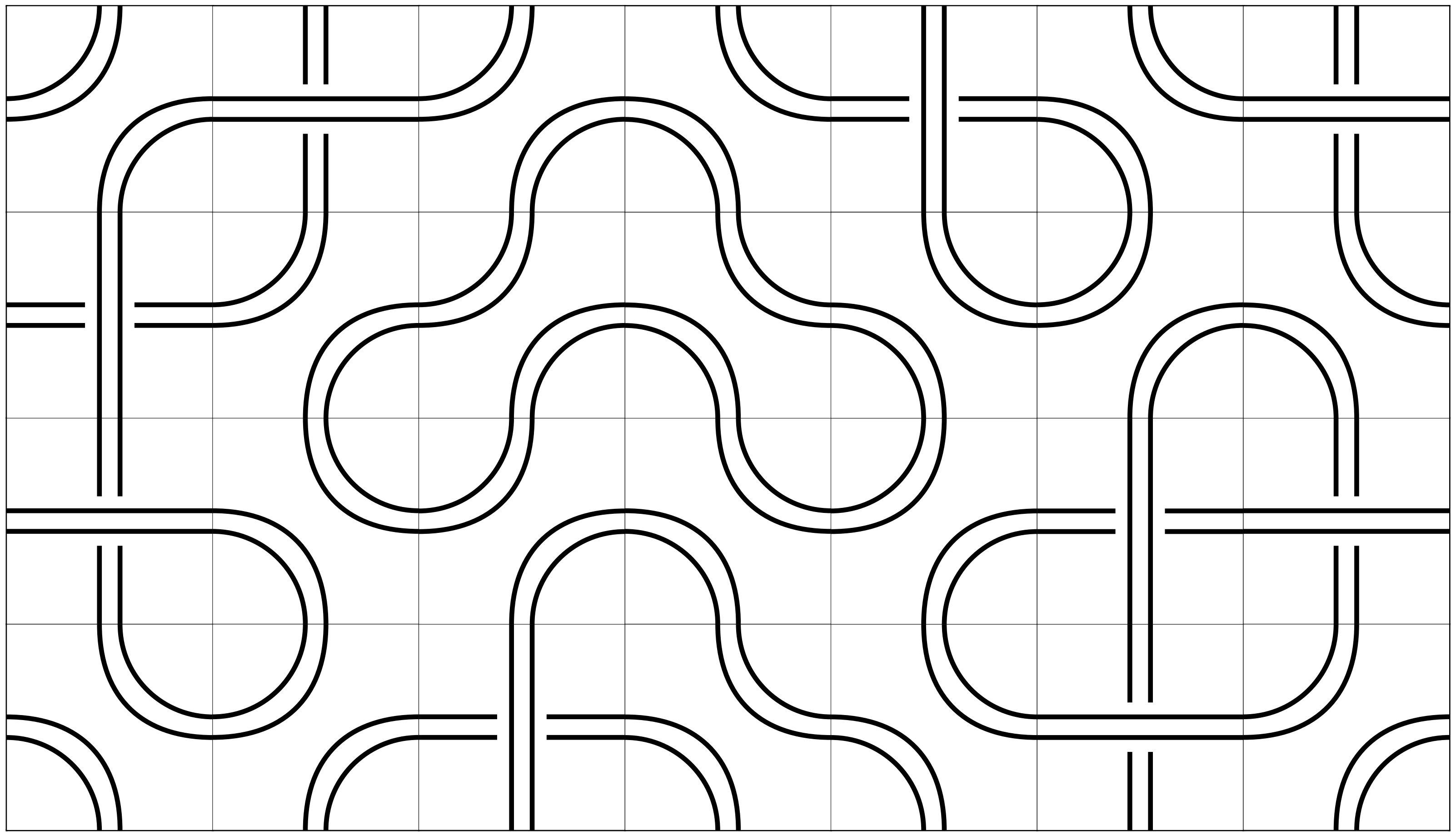
Statistical Mechanics
- Introduction to 2D Lattice Models

Algebra
- Diagrammatic Algebra
- Fendley X
- Face Operators
- Link States and Counting

Conformal Field Theory
- Logarithmic Superconformal Minimal Models
- Finitised Characters
- Kac Tables

2D Lattice Models

- Simplify continuous physical systems by putting them on a lattice.
- Return to continuous system by taking the *continuum limit*.
- Loop models: non-local degrees of freedom.
- Superconformal polymers and percolation.

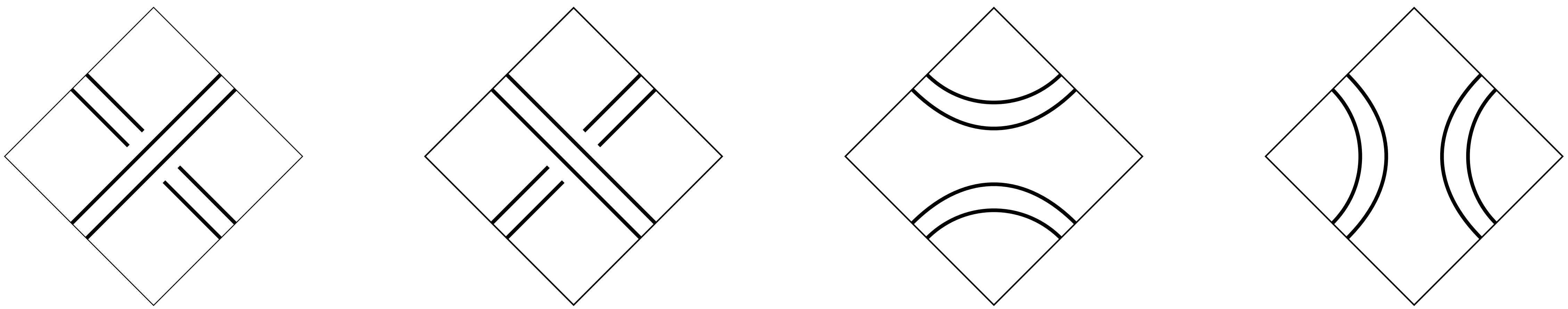


- Over- and under-crossings
- Doubled strands:

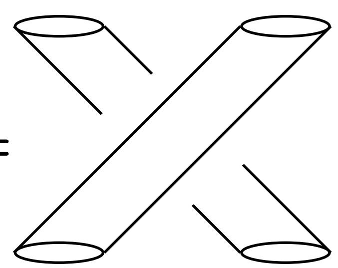
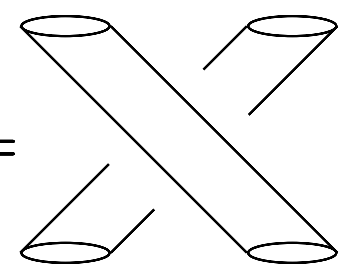
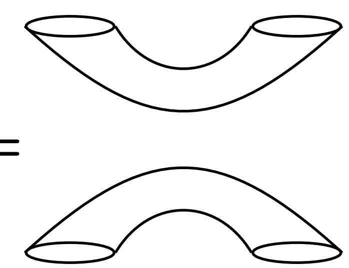
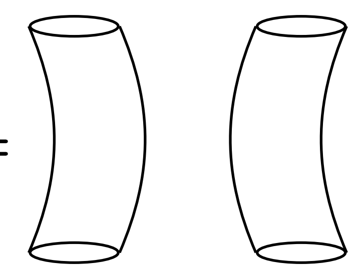
single strands:	logarithmic minimal models	spin-$\frac{1}{2}$
double strands:	superconformal logarithmic minimal models	spin-1

Diagrammatic Algebra

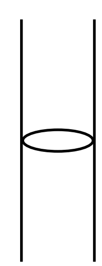
- Tiles on our lattice



- Generators of the fused Temperly-Lieb algebra

$B_j =$  $B_j^{-1} =$  $E_j =$  $I =$ 

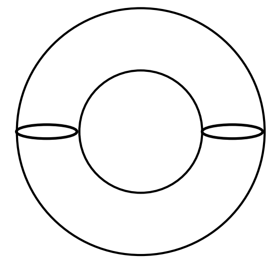
- They are 2×2 fused operators

 $\mapsto$ 

spin-$\frac{1}{2}$ spin-1

Loop Fugacity

- Fused loop fugacity

$\beta_2 =$ 

- Weighting of loops

Superconformal Polymers: $\beta_2 = 0$

Superconformal Percolation: $\beta_2 = 1$

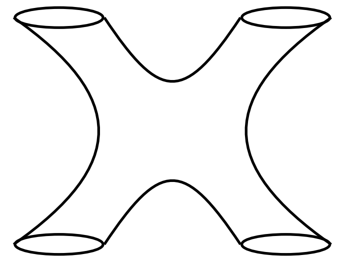
- Can directly see the spin of the models from the loop fugacity

$$\beta_2 = x^2 + 1 + x^{-2} = (x^2)^{+1} + (x^2)^{0} + (x^2)^{-1}$$

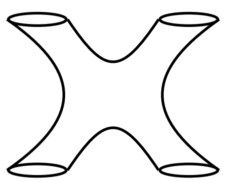
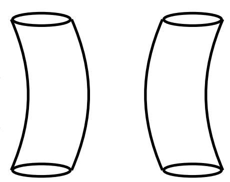
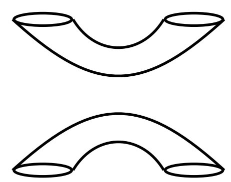
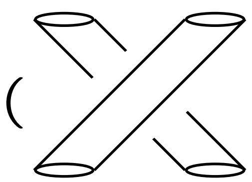
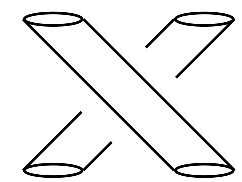
where x is a root of unity.

Fendley X

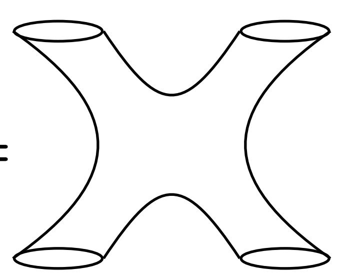
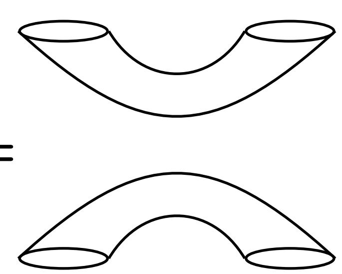
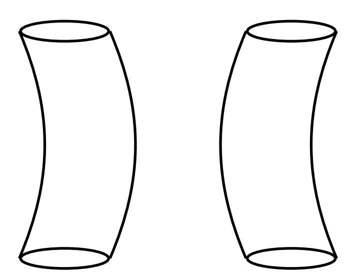
- Introduce new operator,

$X_j =$ 

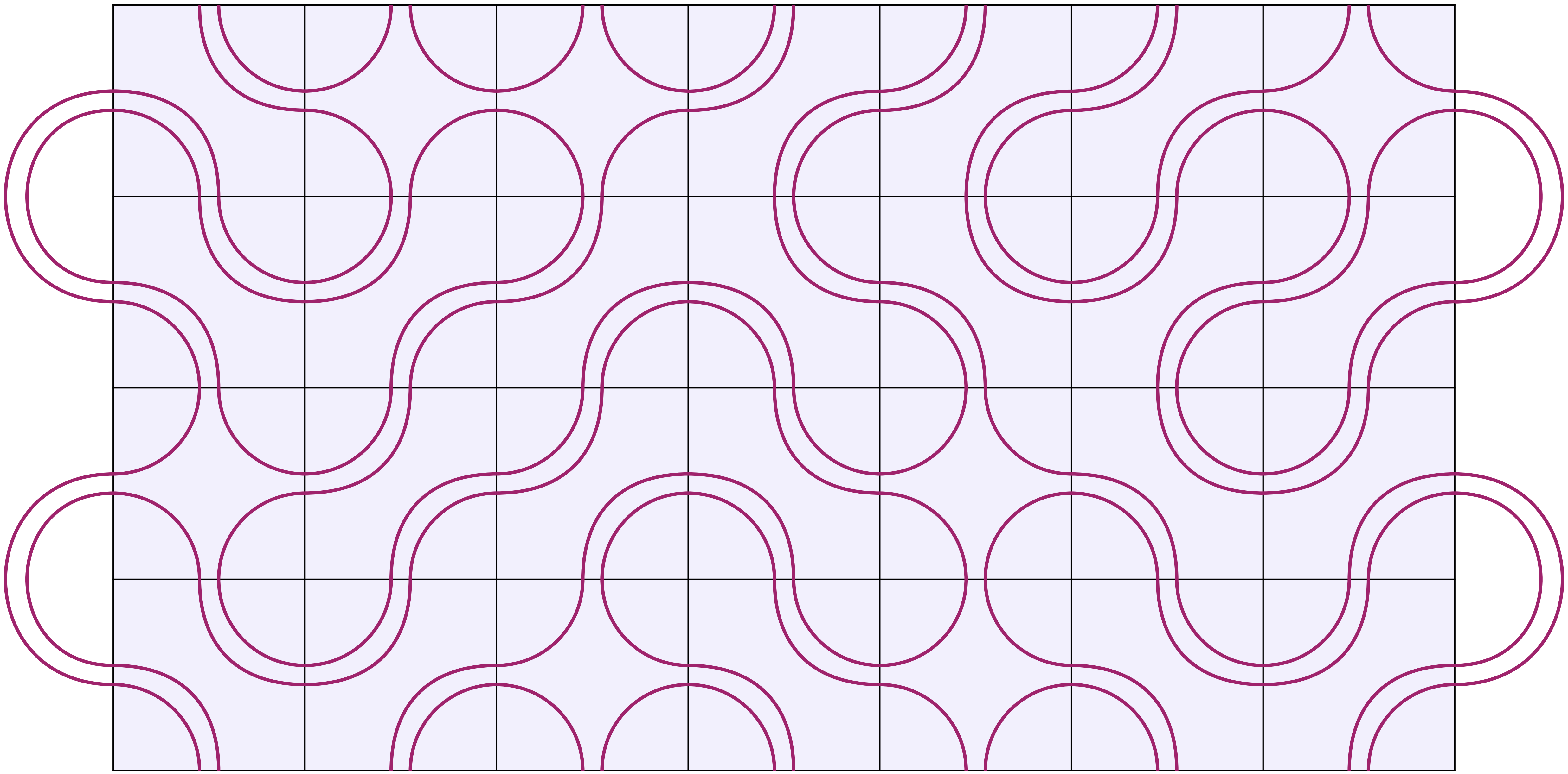
- This new operator can be written in terms of the fused Temperley-Lieb generators

 $= \frac{(x^2 + x^{-2})}{2(x + x^{-1})}($  $+$  $) - \frac{1}{2(x + x^{-1})}($  $+$  $)$

- Rewrite the algebra in terms of

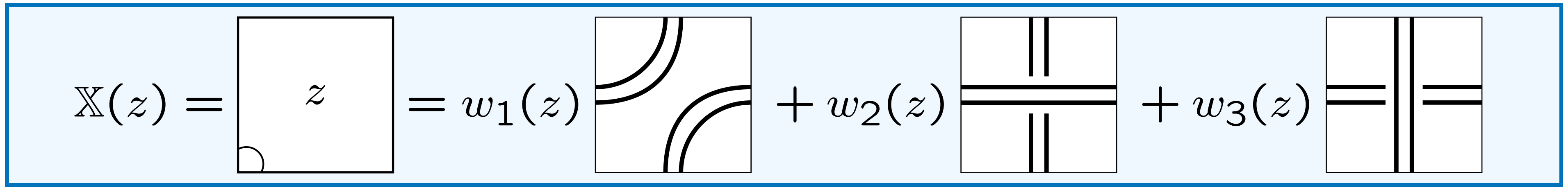
$X_j =$  $E_j =$  $I =$ 

Lattice: X_j, E_j, I



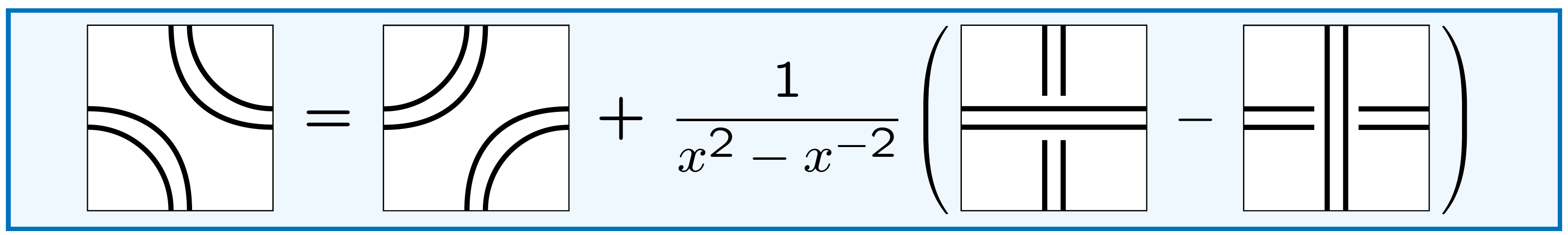
Face Operators (braid)

- The face operators in the braid representation



$$\mathbb{X}_j(z) = w_1(z)I + w_2(z)B_j + w_3(z)B_j^{-1} = I + \frac{z - z^{-1}}{x - x^{-1}}\left(\frac{x^{-1}z}{x^2 - x^{-2}}\,B_j - \frac{xz^{-1}}{x^2 - x^{-2}}\,B_j^{-1}\right)$$

Skein Relation:



$$E_j = I + \frac{B_j - B_j^{-1}}{x^2 - x^{-2}}$$

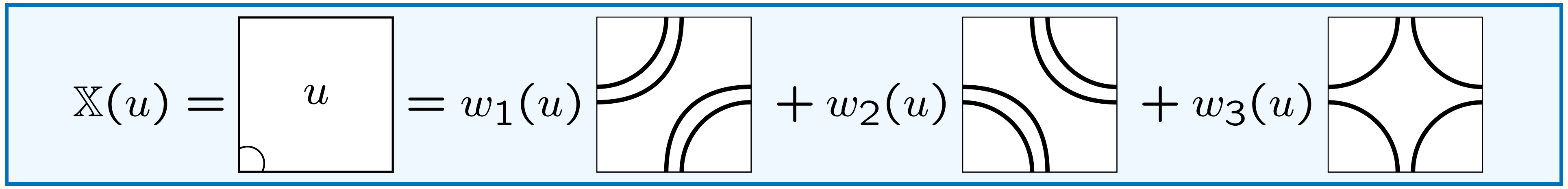
Parameters:

$1 \le p < p'$ coprime integers, $\quad \lambda = \frac{(p'-p)\pi}{p'} =$ crossing parameter

$z = e^{iu}$, $u =$ spectral parameter, $\quad x = e^{i\lambda}$

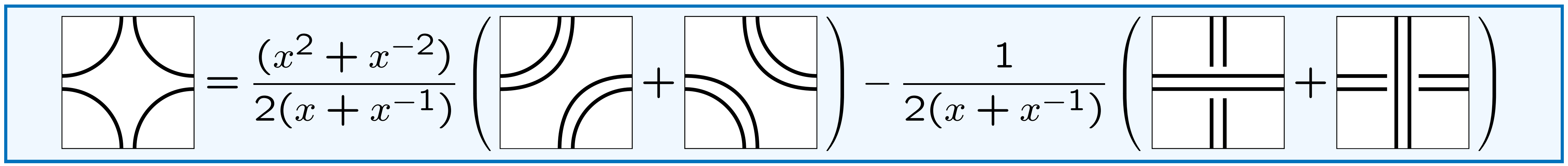
Face Operators (Fendley X)

- The face operators in the Fendley X representation

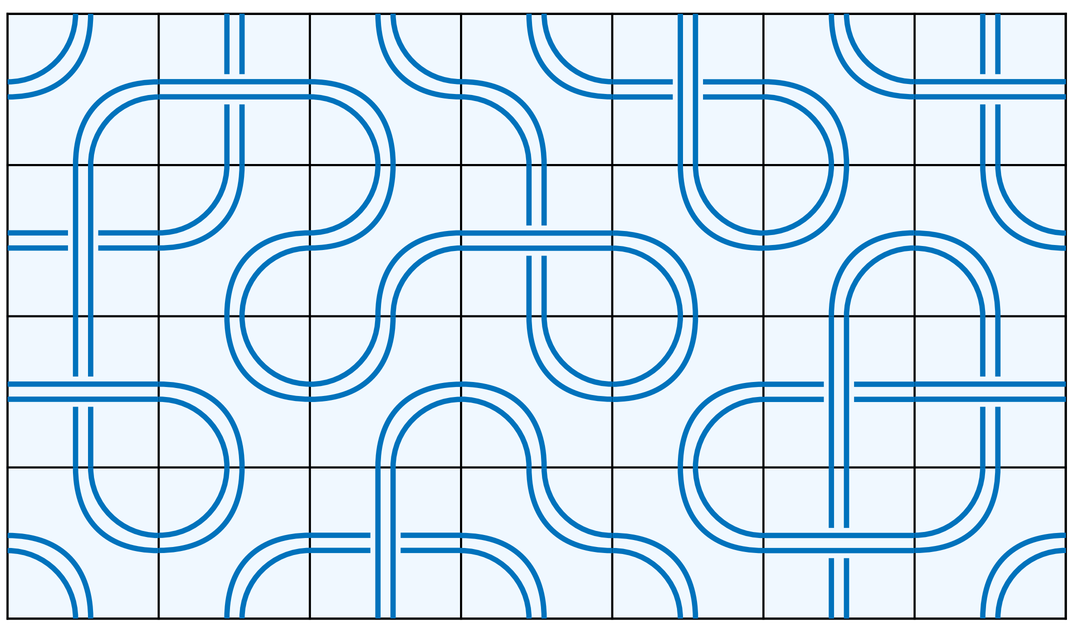


$$\mathbb{X}_j(u) = w_1(u) I + w_2(u) E_j + w_3(u) X_j = \frac{\sin(\lambda-u)\sin(2\lambda-u)}{\sin\lambda\sin 2\lambda} I + \frac{\sin u \sin(\lambda+u)}{\sin\lambda\sin 2\lambda} E_j + \frac{\sin u\sin(\lambda-u)}{\sin^2\lambda} X_j$$

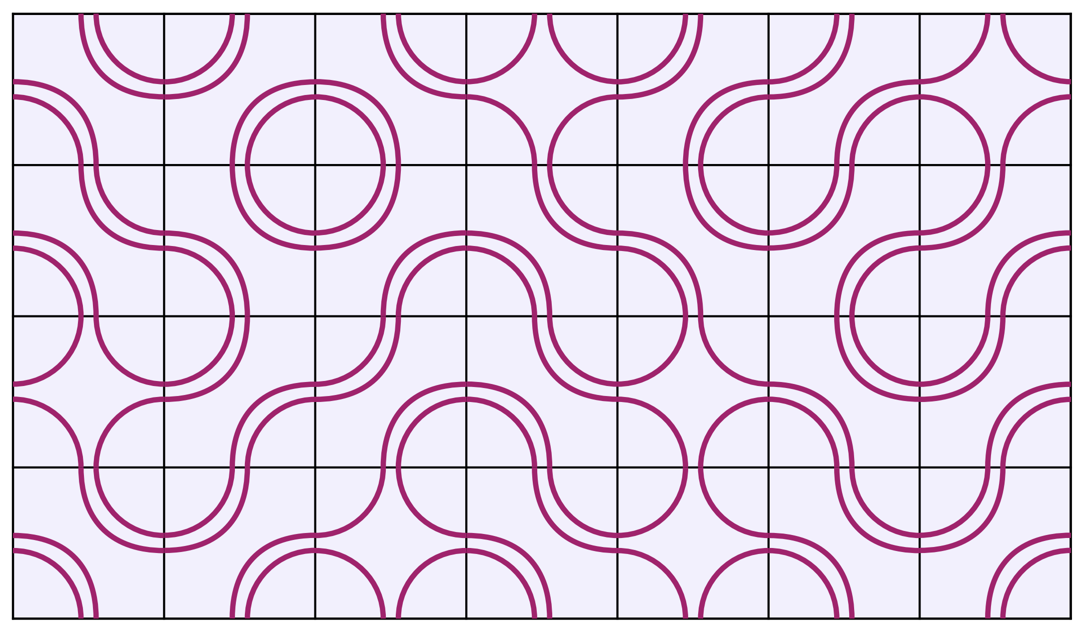
Fendley X:



Superconformal Polymers

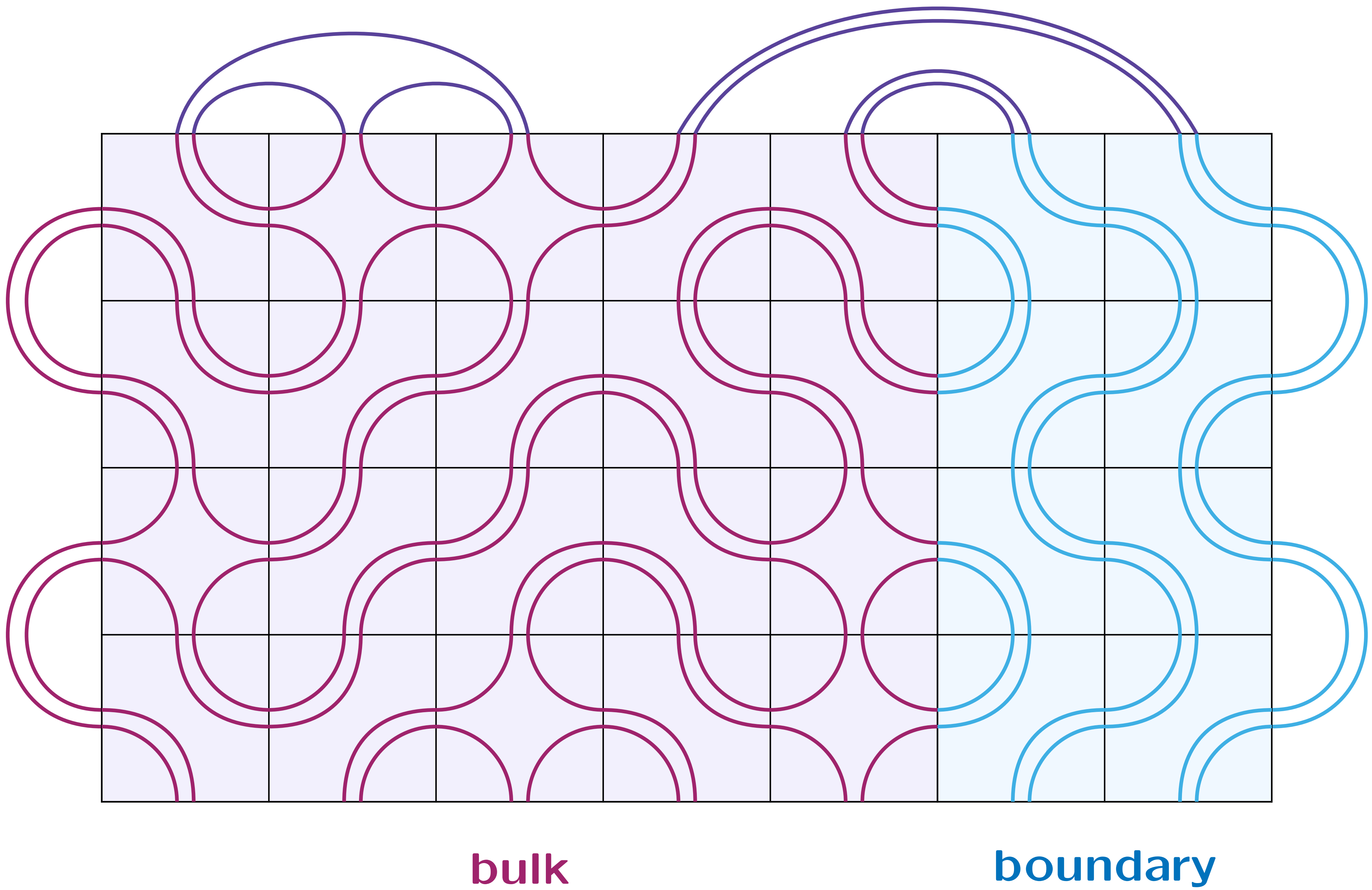


Superconformal Percolation



Link States

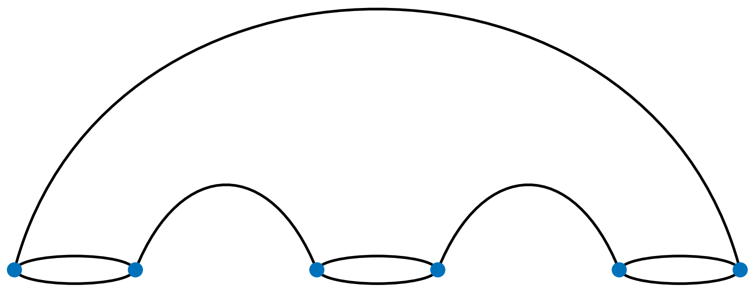
- Link states: states in the quantum mechanical analogue of this system.
- Boundary conditions $(1, s)$: # strands $= s - 1$.



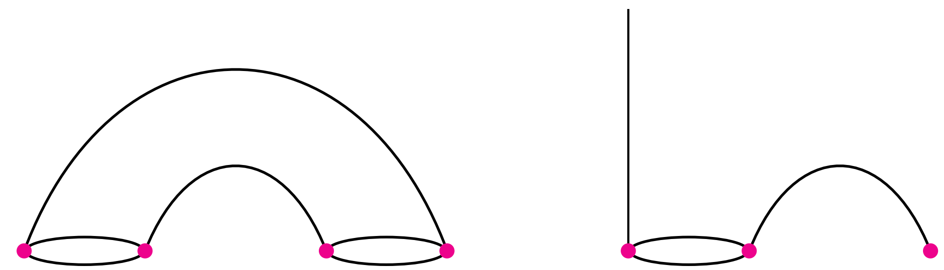
Neveu-Schwarz and Ramond

- Link states can be on and *even* or *odd* number of nodes

Neveu-Schwarz (even)

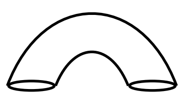


Ramond (odd)

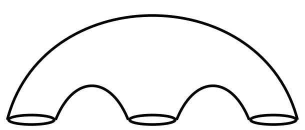


- Link states in the Neveu-Schwarz (NS) sector, $s = 1$

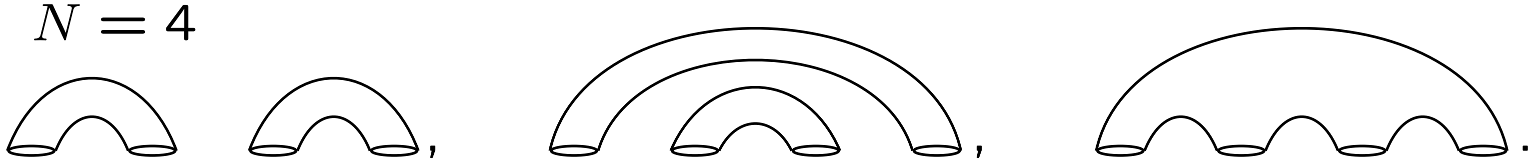
$N = 2$



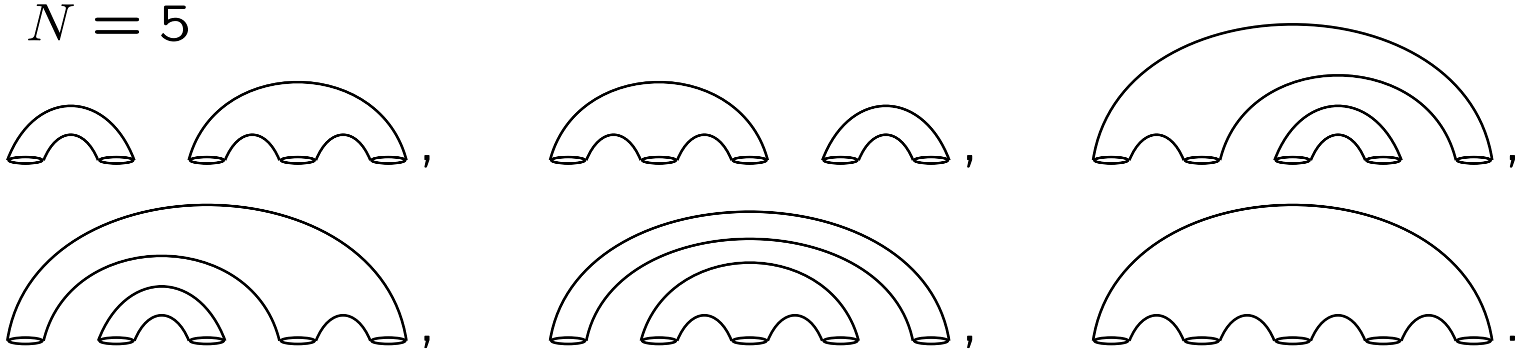
$N = 3$



$N = 4$



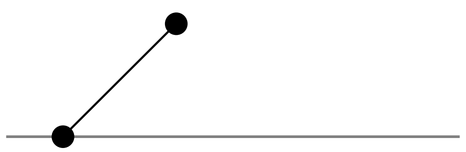
$N = 5$



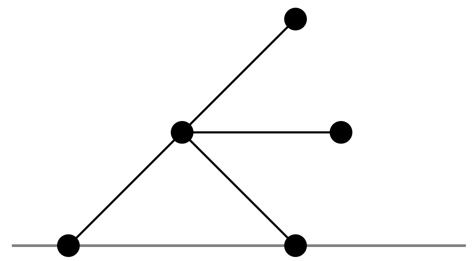
Riordan Numbers

- NS link states for $s = 1$ are counted by *Riordan numbers*, $R_N = 1, 1, 3, 6, 15, 36, \ldots$

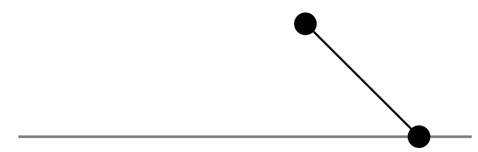
start at ground level



above ground

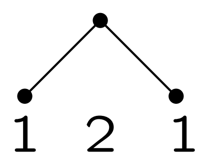


end at ground level

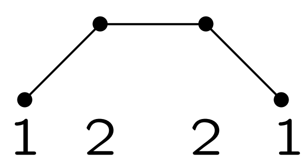


Riordan paths

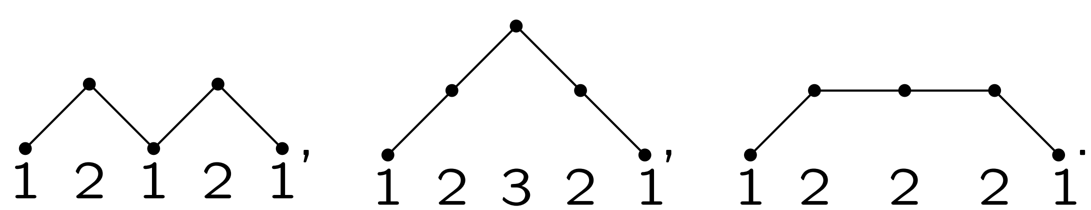
$N = 2$



$N = 3$

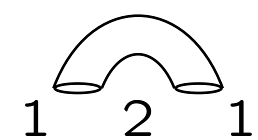


$N = 4$

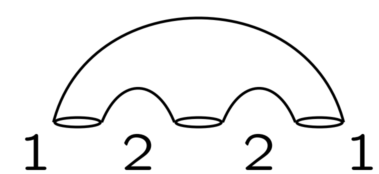


NS link states: $s = 1$

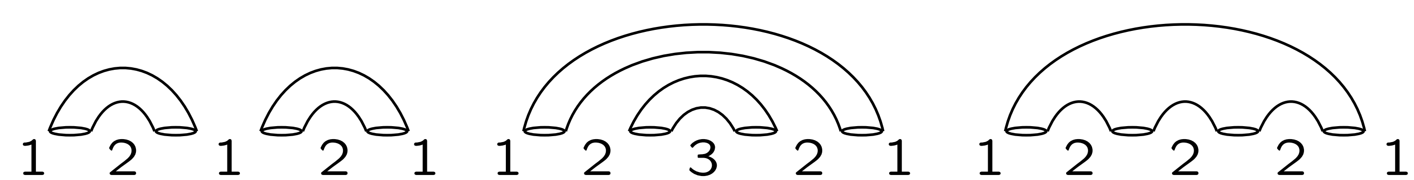
$N = 2$

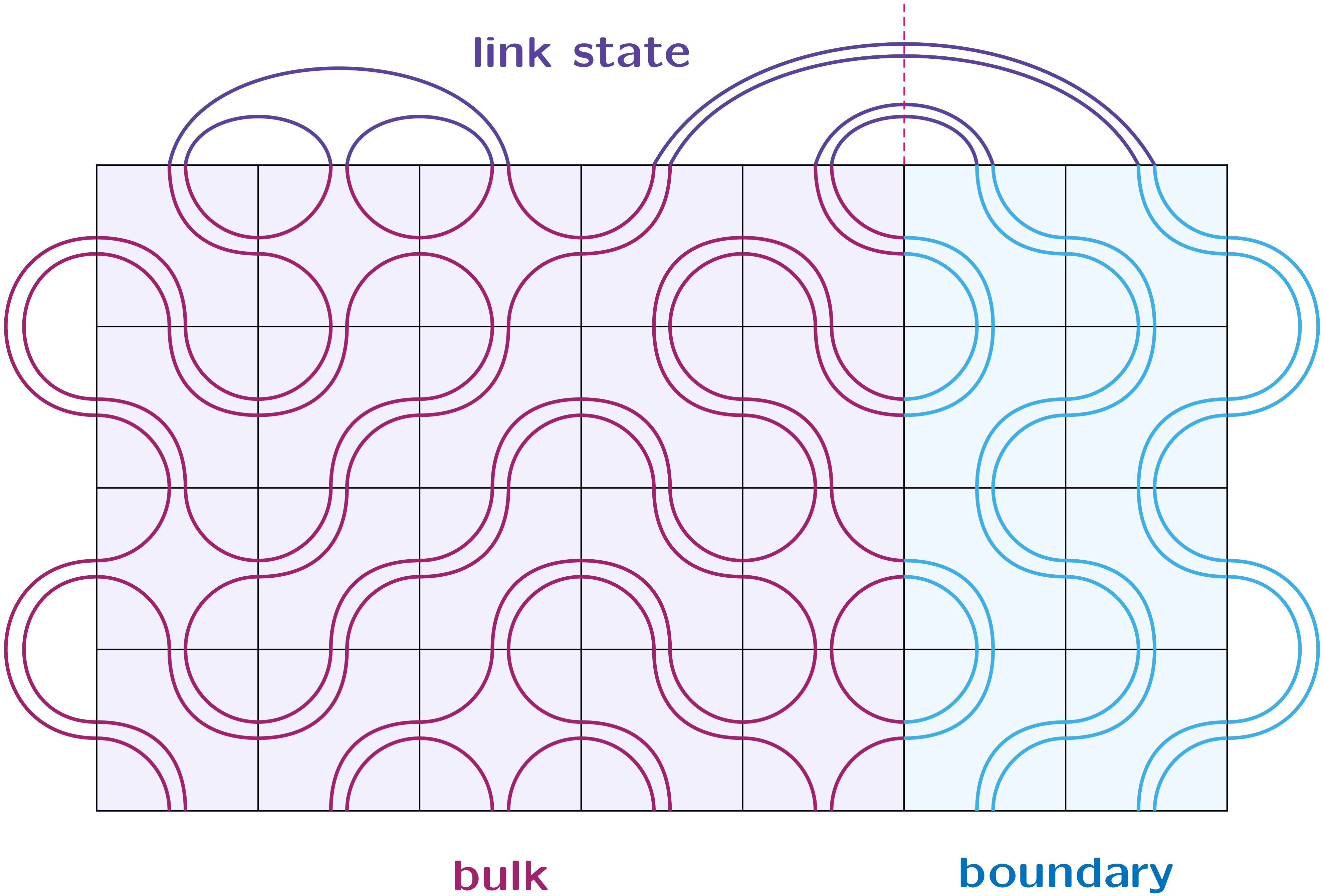


$N = 3$



$N = 4$

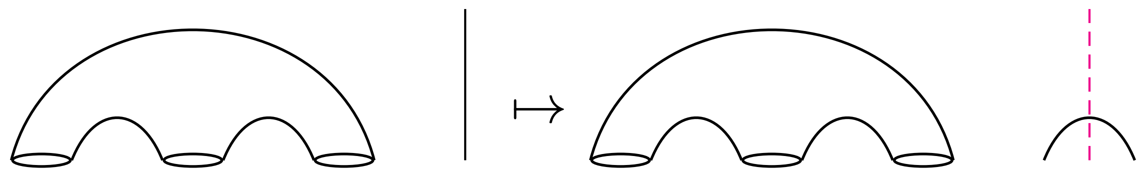




link state
bulk
boundary

Defects

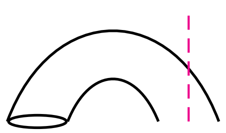
- We introduce link states with *defects*: boundary conditions.



- Link states with *one* defect are on an odd number of nodes

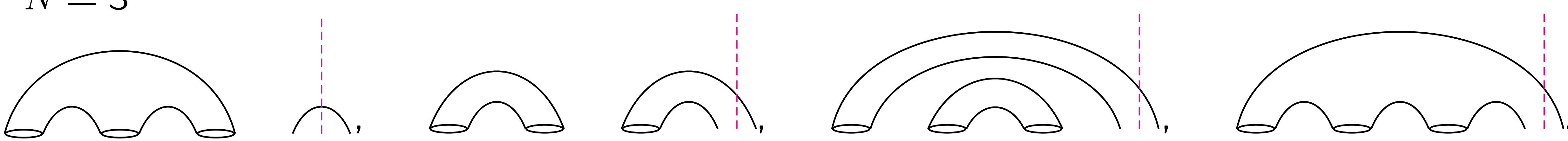
Ramond link states: $s = 2$

$N = 1$



$N = 2$

$N = 3$



- They are counted by *Motzkin* numbers, $M_N = 1, 2, 4, 9, 21, \ldots$

Generalised Riordan Numbers

- Number of NS link states on N nodes with $s-1$ defects is given by

$$R_{N,k} = \binom{N}{k}_2 - \binom{N}{k+1}_2$$

where $2k = s-1$, in terms of *supertrinomials* (spin-1 trinomials)

$$(x+1+x^{-1})^n = \sum_{k=-n}^{n} \binom{n}{k}_2 x^k$$

- Can be written as sums of trinomial coefficients

$$R_{N,k} = \sum_{j=0}^{n} \left(\begin{bmatrix} N \\ \frac{1}{2}(N-j-k), \frac{1}{2}(N-j+k), j \end{bmatrix} - \begin{bmatrix} N \\ \frac{1}{2}(N-j-k-1), \frac{1}{2}(N-j+k+1), j \end{bmatrix} \right)$$

since

$$\binom{n}{k}_2 = \sum_{j=0}^{n} \begin{bmatrix} n \\ \frac{1}{2}(n-j-k), \frac{1}{2}(n-j+k), j \end{bmatrix}$$

where

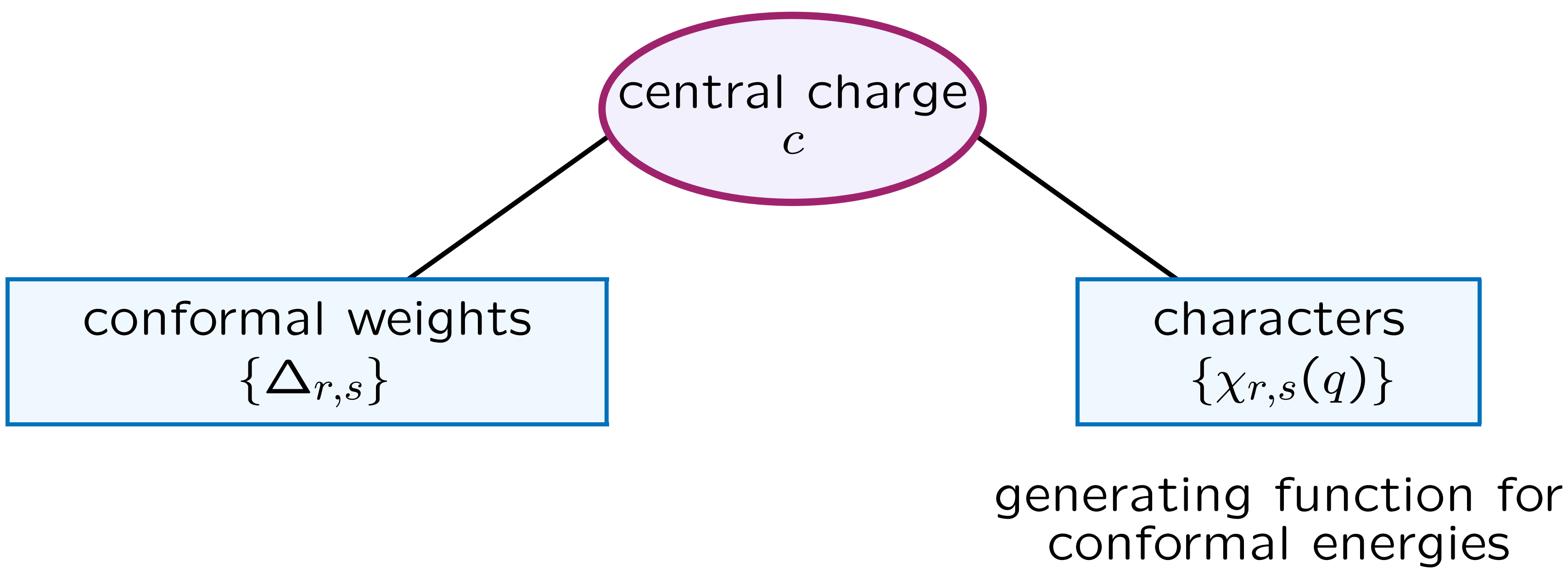
$$\begin{bmatrix} n \\ l, m, n-l-m \end{bmatrix} = \begin{cases} \frac{n!}{l!m!(n-l-m)!}, & l, m, n-l-m \in \mathbb{N}_0 \\ 0, & \text{otherwise} \end{cases}$$

- Number of Ramond link states on N nodes with $s-1$ defects is given by *generalised Motzkin numbers*, $M_{N,k}$, where $2k+1 = s-1$.

Logarithmic Superconformal Minimal Models

- In the continuum limit, our statistical mechanics models correspond to *logarithmic superconformal minimal models*, a type of conformal field theory (CFT).
- The CFT describes the universal properties of the lattice models.
- Every CFT comes with a set of *conformal data*.

Conformal Data



Rational superconformal minimal models $\mathcal{SM}(M, M')$

logarithmic limit → ($M, M' \to \infty$, $\frac{M}{M'} \to \frac{P}{P'}$)

Logarithmic superconformal minimal models $\mathcal{LSM}(P, P')$

Characters

- Characters: generating functions for the conformal energies in the continuum limit.

Neveu-Schwarz: $\ell = 0, 2$, $r+s$ even

$$\chi^{P,P'}_{r,s,\ell}(q) = q^{-\frac{c}{24}+\frac{1}{48}+\Delta^{P,P'}_{r,s}}[c^{\ell}_{m_-}(q) - q^{\frac{rs}{2}} c^{\ell}_{m_+}(q)] = q^{-\frac{c}{24}+\frac{1}{48}+\Delta^{P,P'}_{r,s}}\,\widehat{\chi}_{r,s,\ell}(q)$$

where $m_- = 0, 2 = r - s \bmod 4$ and $m_+ = 0, 2 = r + s \bmod 4$.

Ramond: $\ell = 1$, $r+s$ odd

$$\chi^{P,P'}_{r,s,1}(q) = q^{-\frac{c}{24}+\frac{1}{48}+\Delta^{P,P'}_{r,s}}\,(1 - q^{\frac{rs}{2}})\, c^{1}_{1}(q) = q^{-\frac{c}{24}+\frac{1}{48}+\Delta^{P,P'}_{r,s}}\,\widehat{\chi}_{r,s,1}(q)$$

- In the superconformal case, the string functions c^{ℓ}_{m} are related to $c = \frac{1}{2}$ Ising characters

$$\begin{aligned}
c^{0}_{0}(q) &= c^{2}_{2}(q) = \frac{q^{-1/48}}{2\,(q)_\infty}\left[\prod_{k=1}^{\infty}(1+q^{k-1/2}) + \prod_{k=1}^{\infty}(1-q^{k-1/2})\right]\\
c^{2}_{0}(q) &= c^{0}_{2}(q) = \frac{q^{-1/48}}{2\,(q)_\infty}\left[\prod_{k=1}^{\infty}(1+q^{k-1/2}) - \prod_{k=1}^{\infty}(1-q^{k-1/2})\right]\\
c^{1}_{1}(q) &= \frac{q^{1/24}}{(q)_\infty}\prod_{k=1}^{\infty}(1+q^{k})
\end{aligned}$$

where

$$(q)_\infty = \prod_{k=1}^{\infty}(1-q^{k})$$

Finitised Characters

- Properties of finitised characters

finitised character	$N \to \infty$ ⟶ N even, odd	character
	$q \to 1$ ⟶	link state counting

- In the case $(r,s)=(1,s)$ these are

Neveu-Schwarz: s odd, $\ell = 0, 2$, N even, odd

$$\hat{\chi}^{(N)}_{1,s,\ell}(q) = \sum_{j=0}^{N} q^{\frac{j^2}{2}} \left(\begin{bmatrix} N \\ \frac{1}{2}(N-j-\frac{s-1}{2}), \frac{1}{2}(N-j+\frac{s-1}{2}), j \end{bmatrix}_q - q^{\frac{s}{2}} \begin{bmatrix} N \\ \frac{1}{2}(N-j-\frac{s+1}{2}), \frac{1}{2}(N-j+\frac{s+1}{2}), j \end{bmatrix}_q \right)$$

Ramond: s even, $\ell = 1$, N even and odd

$$\hat{\chi}^{(N)}_{1,s,1}(q) = \sum_{j=0}^{N} q^{\frac{j^2-j}{2}} \left(\begin{bmatrix} N \\ \frac{1}{2}(N-j-\frac{s-2}{2}), \frac{1}{2}(N-j+\frac{s-2}{2}), j \end{bmatrix}_q - q^{\frac{s}{2}} \begin{bmatrix} N \\ \frac{1}{2}(N-j-\frac{s+2}{2}), \frac{1}{2}(N-j+\frac{s+2}{2}), j \end{bmatrix}_q \right)$$

which is written in terms of the q-trinomial

$$\begin{bmatrix} n \\ l, m, n-l-m \end{bmatrix}_q = \begin{cases} \frac{(q)_n}{(q)_l (q)_m (q)_{n-l-m}}, & l, m, n-l-m \in \mathbb{N}_0 \\ 0, & \text{otherwise} \end{cases}$$

polynomial

where

$$(q)_n = \prod_{k=1}^{n} (1-q^k)$$

Constructing Finitised Characters

Neveu-Schwarz $\widehat{\chi}^{(N)}_{r,s,\ell}$

$$\widehat{\chi}^{(N)}_{r,s,\ell} \xrightarrow[N \text{ even, odd}]{N\to\infty} \widehat{\chi}_{r,s,\ell}$$

$$\widehat{\chi}^{(N)}_{r,s,\ell} \xrightarrow{q\to 1} R_{\frac{s-1}{2},N}$$

Ramond $\widehat{\chi}^{(N)}_{r,s,1}$

$$\widehat{\chi}^{(N)}_{r,s,1} \xrightarrow[N \text{ even, odd}]{N\to\infty} \widehat{\chi}_{r,s,1}$$

$$\widehat{\chi}^{(N)}_{r,s,1} \xrightarrow{q\to 1} M_{\frac{s-2}{2},N}$$

- Ansatz for finitised characters

$$\widehat{\chi}^{(N)}(q) = \sum q^{\blacksquare} \left[\cdots\right]_q - q^{\blacksquare} \left[\cdots\right]_q$$

- Can use these to check the conformal weights Δ

$$\chi^{p,p'}_{r,s,\ell}(q) = q^{-\frac{c}{24}+\frac{1}{48}+\Delta^{P,P'}_{r,s}} \lim_{N\to\infty} \widehat{\chi}^{(N)}_{r,s,\ell}(q)$$

Kac Tables

Superconformal polymers

s							
s	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
10	$\frac{31}{16}$	$\frac{1}{2},1$	$-\frac{1}{16}$	$\frac{1}{2},0$	$\frac{15}{16}$	$\frac{5}{2},3$	$\cdots$
9	$\frac{4}{3},\frac{11}{6}$	$\frac{13}{48}$	$\frac{1}{3},-\frac{1}{6}$	$\frac{13}{48}$	$\frac{4}{3},\frac{11}{6}$	$\frac{157}{48}$	$\cdots$
8	$\frac{15}{16}$	$\frac{1}{2},0$	$-\frac{1}{16}$	$\frac{1}{2},1$	$\frac{31}{16}$	$\frac{9}{2},4$	$\cdots$
7	$1,\frac{1}{2}$	$-\frac{1}{16}$	$0,\frac{1}{2}$	$\frac{15}{16}$	$3,\frac{5}{2}$	$\frac{79}{16}$	$\cdots$
6	$\frac{13}{48}$	$-\frac{1}{6},\frac{1}{3}$	$\frac{13}{48}$	$\frac{11}{6},\frac{4}{3}$	$\frac{157}{48}$	$\frac{35}{6},\frac{19}{3}$	$\cdots$
5	$0,\frac{1}{2}$	$-\frac{1}{16}$	$1,\frac{1}{2}$	$\frac{31}{16}$	$4,\frac{9}{2}$	$\frac{111}{16}$	$\cdots$
4	$-\frac{1}{16}$	$\frac{1}{2},0$	$\frac{15}{16}$	$\frac{5}{2},3$	$\frac{79}{16}$	$\frac{17}{2},8$	$\cdots$
3	$\frac{1}{3},-\frac{1}{6}$	$\frac{13}{48}$	$\frac{4}{3},\frac{11}{6}$	$\frac{157}{48}$	$\frac{19}{3},\frac{35}{6}$	$\frac{445}{48}$	$\cdots$
2	$-\frac{1}{16}$	$\frac{1}{2},1$	$\frac{31}{16}$	$\frac{9}{2},4$	$\frac{111}{16}$	$\frac{21}{2},11$	$\cdots$
1	$0,\frac{3}{2}$	$\frac{15}{16}$	$3,\frac{5}{2}$	$\frac{79}{16}$	$8,\frac{17}{2}$	$\frac{191}{16}$	$\cdots$
	1	2	3	4	5	6	r

$\mathcal{LSM}(1,3): c=-\frac{5}{2},\ \Delta^{1,3}_{r,s,\ell}$

$\beta_2 = 0$

Superconformal percolation

s	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
10	4	$\frac{35}{16},\frac{43}{16}$	1	$\frac{11}{16},\frac{3}{16}$	0	$\frac{3}{16},\frac{11}{16}$	$\cdots$
9	$3,\frac{7}{2}$	$\frac{25}{16}$	$1,\frac{1}{2}$	$\frac{1}{16}$	$0,\frac{1}{2}$	$\frac{9}{16}$	$\cdots$
8	$\frac{9}{4}$	$\frac{23}{16},\frac{15}{16}$	$\frac{1}{4}$	$-\frac{1}{16},\frac{7}{16}$	$\frac{1}{4}$	$\frac{23}{16},\frac{15}{16}$	$\cdots$
7	$2,\frac{3}{2}$	$\frac{9}{16}$	$0,\frac{1}{2}$	$\frac{1}{16}$	$1,\frac{1}{2}$	$\frac{25}{16}$	$\cdots$
6	1	$\frac{3}{16},\frac{11}{16}$	0	$\frac{11}{16},\frac{3}{16}$	1	$\frac{35}{16},\frac{43}{16}$	$\cdots$
5	$\frac{1}{2},1$	$\frac{1}{16}$	$\frac{1}{2},0$	$\frac{9}{16}$	$\frac{3}{2},2$	$\frac{49}{16}$	$\cdots$
4	$\frac{1}{4}$	$\frac{7}{16},-\frac{1}{16}$	$\frac{1}{4}$	$\frac{15}{16},\frac{23}{16}$	$\frac{9}{4}$	$\frac{71}{16},\frac{63}{16}$	$\cdots$
3	$\frac{1}{2},0$	$\frac{1}{16}$	$\frac{1}{2},1$	$\frac{25}{16}$	$\frac{7}{2},3$	$\frac{81}{16}$	$\cdots$
2	0	$\frac{3}{16},\frac{11}{16}$	1	$\frac{43}{16},\frac{35}{16}$	4	$\frac{99}{16},\frac{107}{16}$	$\cdots$
1	$0,\frac{3}{2}$	$\frac{9}{16}$	$2,\frac{3}{2}$	$\frac{49}{16}$	$5,\frac{11}{2}$	$\frac{121}{16}$	$\cdots$
	1	2	3	4	5	6	r

$\mathcal{LSM}(2,4): c=0,\ \Delta^{2,4}_{r,s,\ell}$

$\beta_2 = 1$

$$\chi^{P,P'}_{r,s,\ell}(q) = q^{-\frac{c}{24}+\frac{1}{48}+\Delta^{P,P'}_{r,s}} \lim_{N\to\infty} \hat{\chi}^{(N)}_{r,s,\ell}(q)$$