Generalized Verma and Wakimoto Modules

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What is an affine Kac-Moody algebra?
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Central extensions of loop algebras.
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Loop algebra:

$\mathfrak{g}((t)) := \mathfrak{g} \oplus \mathbb{C}((t))$, 

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For every non-degenerate bilinear form $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$, one can define an exact sequence

$$0 \to \mathbb{C}.1 \to \widehat{\mathfrak{g}}_\kappa \to \mathfrak{g}((t)) \to 0$$

with the two cocycle defined by

$$x \otimes f(t), y \otimes g(t) \mapsto -\kappa(x, y) \cdot \text{Res}_{t=0} f \, dg.$$
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The level $\kappa_{\text{crit}} = -\frac{1}{2} \kappa_{\text{Kil}}$ is the critical level.
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All modules in category $O$ are smooth. However, the category of smooth modules is much bigger.
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Why do I care?

1. Kac-Moody algebras and groups have a characteristic $p$ cousin. Smooth representations of the latter objects carry important number-theoretic information, elucidated by the Langlands conjecture.

2. Smooth representations of affine Kac-Moody algebras play a central role in the Beilinson and Drinfeld’s approach to the geometric Langlands program.
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\[ \mathfrak{h} \text{ Cartan, } \mathfrak{b} \text{ Borel containing } \mathfrak{h} \]
\[ \mathfrak{h} \text{ Cartan, } b \text{ Borel containing } \mathfrak{h} \]
\[ \mathfrak{g}_n := \mathfrak{g} \otimes \mathbb{C}[t]/t^n, \quad b_n = b \otimes \mathbb{C}[t]/t^n, \quad \mathfrak{h}_n = \mathfrak{h} \otimes \mathbb{C}[t]/t^n. \]
- $\mathfrak{h}$ Cartan, $\mathfrak{b}$ Borel containing $\mathfrak{h}$
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- $\mathfrak{b}_n$ the inverse image of $\mathfrak{b}_n$ under the map $\mathfrak{g}[t] \to \mathfrak{g}_n$. 

A character $\Lambda : \mathfrak{h}_n \to \mathbb{C}$ defines a character of $\hat{\mathfrak{b}}_n$ via the maps $\hat{\mathfrak{b}}_n \to \mathfrak{b}_n \to \mathfrak{h}_n \to \mathbb{C}$.

Generalised Verma module $M^\kappa(\Lambda) := \text{Ind}_{\hat{\mathfrak{g}}_{\kappa} \hat{\mathfrak{b}}_n}^{\hat{\mathfrak{g}}_{\kappa}} C^1(\Lambda)$. 

1. What is the endomorphism algebra of $M^\kappa(\Lambda)$?
2. How does the centre (at the critical level) act on $M^\kappa(\Lambda)$?

To answer these questions, we relate Verma modules to Wakimoto modules.
\begin{itemize}
  \item $\mathfrak{h}$ Cartan, $\mathfrak{b}$ Borel containing $\mathfrak{h}$
  \item $g_n := g \otimes \mathbb{C}[t]/t^n$, $b_n = b \otimes \mathbb{C}[t]/t^n$, $h_n = h \otimes \mathbb{C}[t]/t^n$.
  \item $\hat{b}_n$ the inverse image of $b_n$ under the map $g[t] \to g_n$.
  \item A character $\Lambda : h_n \to \mathbb{C}$ defines a character of $\hat{b}_n$ via the maps
    \[ \hat{b}_n \to b_n \to h_n \xrightarrow{\Lambda} \mathbb{C}. \]
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- Generalised Verma module
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  \mathcal{M}_\kappa(\Lambda) := \text{Ind}^{\hat{\mathfrak{g}}_\kappa}_{\mathfrak{b}_n \oplus \mathbb{C} \cdot 1}(\Lambda).
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$$\mathcal{M}_\kappa(\Lambda) := \text{Ind}_{\hat{\mathfrak{g}}_\kappa}^{\hat{\mathfrak{g}}_\kappa}(\Lambda).$$

- $n = 1$ : usual Verma
Smooth representation of affine Kac-Moody algebras

Generalized Verma Modules

Generalized Wakimoto Modules

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- Generalised Verma module
  \[ M_{\kappa}(\Lambda) := \text{Ind}^{\hat{\mathfrak{g}}_\kappa}_{\hat{\mathfrak{b}}_n \oplus \mathbb{C}.1}(\Lambda). \]
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- $n > 1$, not in category $\mathcal{O}$, but smooth.
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\[
\begin{align*}
g_n & := g \otimes \mathbb{C}[t]/t^n, \\
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\( n > 1 \), not in category \( \mathcal{O} \), but smooth.

Basic questions:

- What is the endomorphism algebra of \( \mathcal{M}_\kappa(\Lambda) \)?
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The Weyl algebra $\mathcal{A}$ is the associative algebra generated by $a_n$ and $a_n^*$, $n \in \mathbb{Z}$, subject to

$$[a_n, a_m^*] = \delta_{n,-m}, \quad [a_n, a_m] = [a_n^*, a_m^*] = 0.$$
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$\kappa$ restricts to a non-degenerate form $\mathfrak{h} \times \mathfrak{h} \rightarrow \mathbb{C}$. So we can define the algebra

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Wakimoto-Feigin-Frenkel free field realisation:
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Input: a module $L$ over $\widehat{\mathfrak{h}}_\kappa$ and a module $N$ over $\mathcal{A}$.
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The critical shift is a "quantum correction" arising because of normal ordering of fields.
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We have a character $\mathfrak{h}[[t]] \to \mathfrak{h}_n \xrightarrow{\Lambda} \mathbb{C}$.

$L := \text{Ind}_{\mathfrak{h}[[t]] \oplus \mathbb{C}.1}^{\mathfrak{h}_n} (\Lambda)$. 

$N$ is generated by a vacuum vector $|0\rangle$ subject to $a_m |0\rangle = 0$, $m \geq n$ and $a^*_m |0\rangle = 0$, $m \geq 1 - n$.

These modules appear in Fridan-Martinec-Shankar's "Conformal invariance, supersymmetry, and string theory". In mathematics literature, it seems only the case $n = 0$ and $n = 1$ are considered.
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Generalised Wakimoto module:

$W_{\kappa + \kappa_{\text{crit}}}^{(\Lambda)} := L \otimes N$.
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Relationship between generalised Verma and Wakimoto modules:
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- Proposition: For every $\Lambda$, there exists a nontrivial morphism $\mathcal{M}_\kappa(\Lambda) \rightarrow \mathcal{W}_\kappa(\Lambda)$. 

Thank you!

Masoud Kamgarpour
The University of Queensland
Relationship between generalised Verma and Wakimoto modules:

- Proposition: For every $\Lambda$, there exists a nontrivial morphism $\mathcal{M}_\kappa(\Lambda) \to \mathcal{W}_\kappa(\Lambda)$.

- Theorem: The centre at the critical level acts by the same quotient on $\mathcal{M}_{\kappa_{\text{crit}}}(\Lambda)$ and $\mathcal{W}_{\kappa_{\text{crit}}}(\Lambda)$.

Evidence for the conjecture:

1. If $n = 1$, this is a theorem of Frenkel.
2. In characteristic $p$, this is a theorem of Bernstein, Bushnell, Kutzko, Roche, ...
- Relationship between generalised Verma and Wakimoto modules:

- Proposition: For every \( \Lambda \), there exists a nontrivial morphism \( M_\kappa (\Lambda) \rightarrow W_\kappa (\Lambda) \).

- Theorem: The centre at the critical level acts by the same quotient on \( M_{\kappa_{\text{crit}}} (\Lambda) \) and \( W_{\kappa_{\text{crit}}} (\Lambda) \).

- Conjecture: For “generic” values of \( \Lambda \), the morphism \( M_{\kappa_{\text{crit}}} (\Lambda) \rightarrow W_{\kappa_{\text{crit}}} (\Lambda) \) is an isomorphism.
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  - Proposition: For every $\Lambda$, there exists a nontrivial morphism $M_\kappa(\Lambda) \to W_\kappa(\Lambda)$.
  - Theorem: The centre at the critical level acts by the same quotient on $M_{\kappa_{\text{crit}}}(\Lambda)$ and $W_{\kappa_{\text{crit}}}(\Lambda)$.
  - Conjecture: For “generic” values of $\Lambda$, the morphism $M_{\kappa_{\text{crit}}}(\Lambda) \to W_{\kappa_{\text{crit}}}(\Lambda)$ is an isomorphism.
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Thank you!