Properties of the Bethe Ansatz equations for Richardson-Gaudin models

Inna Lukyanenko, Phillip Isaac, Jon Links

Centre for Mathematical Physics,
School of Mathematics and Physics,
The University of Queensland

Some History

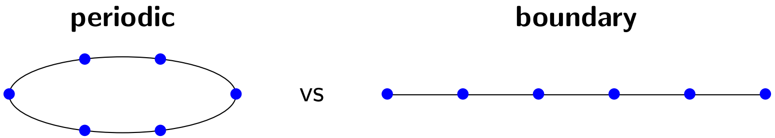
- **Quantum Inverse Scattering Method (QISM)** for twisted periodic boundary conditions [Faddeev, Kulish, Sklyanin, Takhtajan 1979].

- **BCS model** of superconductivity [Bardeen, Cooper and Schrieffer 1957].
  - Solved for the **rational case** [Richardson 1963].
  - **Integrals of motion** for the Richardson model [Cambiaggio et al. 1997].
  - **Eigenvalues** for the Richardson model [Sierra 2000].
  - Generalized to the **trigonometric case** using Gaudin's method [Amico et al. 2001], [Dukelsky et al. 2001].
  - Reformulated through the **quasi-classical limit** of QISM [Zhou et al. 2002], [von Delft and Poghossian 2002].

- **Some extensions:** [Ovchinnikov 2003], [Dunning and Links 2004], [Ibañez et al. 2009], [Skrypnyk 2009], [Dukelsky et al. 2010, 2011], [Links and Marquette 2013].

Some History

- **Boundary QISM (BQISM)** for open-boundary conditions, for the case of XXZ spin chain [Sklyanin 1988].



- What is the effect of the "boundary" for **Richardson-Gaudin models**?
  - **Gaudin magnet** with boundary [Hikami 1995].
  - **Quasi-classical limit** of the BQISM [Di Lorenzo et al. 2002].
  - **Generalized** Gaudin systems [Skrypnyk 2006, 2007, 2010].
  - **Trigonometric** Gaudin model [Cirilio António et al. 2013].

Outline

In the quasi-classical limit (QC)

$$
\begin{tikzcd}
\textbf{QC trig. BQISM} \arrow[r, shift left, "\text{attenuated limit}"] \arrow[d, shift right, "\text{rational limit}"'] & \textbf{QC trig. QISM} \arrow[l, shift left, "\text{variable change}"] \arrow[d, "\text{rational limit}"'] \\
\textbf{QC rat. BQISM} \arrow[u, shift right, "\text{variable change}"'] \arrow[r, "\text{attenuated limit}"] & \textbf{QC rat. QISM}
\end{tikzcd}
$$

R-matrix

R-**matrix** is an operator $R(u) \in \mathrm{End}(V \otimes V)$ ($V \cong \mathbb{C}^2$, $u \in \mathbb{C}$) satisfying the **Yang-Baxter equation (YBE)** in $\mathrm{End}(V \otimes V \otimes V)$:

$$\boxed{R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v)}$$

- **Rational solution** ($\eta \in \mathbb{C}, \quad P(u \otimes v) = v \otimes u, \ \forall u, v \in V$)

$$R^{rat}(u) = \frac{1}{u+\eta}(uI \otimes I + \eta P) = \frac{1}{u+\eta}\begin{pmatrix} u+\eta & 0 & 0 & 0 \\ 0 & u & \eta & 0 \\ 0 & \eta & u & 0 \\ 0 & 0 & 0 & u+\eta \end{pmatrix}.$$

- **Trigonometric solution**

$$R^{trig}(u) = \frac{1}{\sinh(u+\eta)}\begin{pmatrix} \sinh(u+\eta) & 0 & 0 & 0 \\ 0 & \sinh u & \sinh\eta & 0 \\ 0 & \sinh\eta & \sinh u & 0 \\ 0 & 0 & 0 & \sinh(u+\eta) \end{pmatrix}.$$

Rational limit $\lim_{\nu \to 0} \frac{\sinh(\nu x)}{\nu} = x :$ $\boxed{\textbf{Trigonometric} \longrightarrow \textbf{Rational}}$

QISM [Faddeev et al. 1979]

Monodromy matrix $\in \mathrm{End}\left(V_a \otimes V^{\otimes \mathcal{L}}\right)$ (where $V_a = \mathbb{C}^2$ is the auxiliary space, $V^{\otimes \mathcal{L}} = \underbrace{V \otimes V \otimes \ldots \otimes V}_{\mathcal{L} \text{ times}}$ is the quantum space, $\mathcal{L} \in \mathbb{N}$)

$$T_a(u) := \begin{pmatrix} e^{-\eta\gamma} & 0 \\ 0 & e^{\eta\gamma} \end{pmatrix} R_{a\mathcal{L}}(u-\varepsilon_{\mathcal{L}})\ldots R_{a2}(u-\varepsilon_2)R_{a1}(u-\varepsilon_1) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$$

Transfer matrix $t(u) := tr_a(T_a(u)) = A(u) + D(u) \in \mathrm{End}(V^{\otimes \mathcal{L}})$:

$$\boxed{[t(u), t(v)] = 0 \quad \forall u, v \in \mathbb{C}}$$

Then $t(u)$ generates a set of **mutually commuting** operators $\{C_j\}$:

$$t(u) = \sum_{j=-\infty}^{\infty} C_j u^j .$$

Take any function of $\{C_j\}$ as the Hamiltonian. Then $\{C_j\}$ are mutually commuting **integrals of motion**.

Algebraic Bethe Ansatz

Start with a **reference state** $\Omega \in V^{\otimes \mathcal{L}}$:

$$B(u)\Omega = 0, \;\; A(u)\Omega = a(u)\Omega, \;\; D(u)\Omega = d(u)\Omega, \;\; C(u)\Omega \neq 0.$$

Then (for the **trigonometric** R-matrix)

$$\Phi(v_1, ..., v_N) = C(v_1)...C(v_N)\Omega$$

is an **eigenstate** of $t(u)$ with the **eigenvalue**

$$\Lambda(u, v_1, ..., v_N) = a(u) \prod_{k=1}^{N} \frac{\sinh(u - v_k + \eta)}{\sinh(u - v_k)} + d(u) \prod_{k=1}^{N} \frac{\sinh(u - v_k - \eta)}{\sinh(u - v_k)},$$

if $\Phi \neq 0$ and v's satisfy the **Bethe Ansatz equations (BAE)**

$$\boxed{\frac{a(v_k)}{d(v_k)} = \prod_{i \neq k}^{N} \frac{\sinh(v_k - v_i - \eta)}{\sinh(v_k - v_i + \eta)}, \quad k = 1, ..., N}$$

For the **rational** R-matrix: $\sinh(x) \to x$.

Algebraic Bethe Ansatz

Start with a **reference state** $\Omega \in V^{\otimes \mathcal{L}}$:

$$B(u)\Omega = 0, \ A(u)\Omega = a(u)\Omega, \ D(u)\Omega = d(u)\Omega, \ C(u)\Omega \neq 0.$$

Then (for the **trigonometric** R-matrix)

$$\Phi(v_1, ..., v_N) = C(v_1)...C(v_N)\Omega$$

is an **eigenstate** of $t(u)$ with the **eigenvalue**

$$\Lambda(u, v_1, ..., v_N) = a(u) \prod_{k=1}^{N} \frac{\sinh(u - v_k + \eta)}{\sinh(u - v_k)} + d(u) \prod_{k=1}^{N} \frac{\sinh(u - v_k - \eta)}{\sinh(u - v_k)},$$

if $\Phi \neq 0$ and v's satisfy the **Bethe Ansatz equations (BAE)**

$$\boxed{\frac{a(v_k)}{d(v_k)} = \prod_{i \neq k}^{N} \frac{\sinh(v_k - v_i - \eta)}{\sinh(v_k - v_i + \eta)}, \quad k = 1, ..., N}$$

For the **rational** R-matrix: $\sinh(x) \to x$.

Quasi-classical limit (QC)

Derive the expressions for $a(u)$ and $d(u)$ for $\Omega = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^{\otimes \mathcal{L}}$:

$$a(u) = e^{-\eta\gamma} \prod_{l=1}^{\mathcal{L}} \frac{\sinh(u - \varepsilon_l - \eta/2)}{\sinh(u - \varepsilon_l)}, \quad d(u) = e^{\eta\gamma} \prod_{l=1}^{\mathcal{L}} \frac{\sinh(u - \varepsilon_l + \eta/2)}{\sinh(u - \varepsilon_l)}.$$

Then the **BAE**:

$$e^{-2\eta\gamma} \prod_{l=1}^{\mathcal{L}} \frac{\sinh(v_k - \varepsilon_l - \eta/2)}{\sinh(v_k - \varepsilon_l + \eta/2)} = \prod_{i \neq k}^{N} \frac{\sinh(v_k - v_i - \eta)}{\sinh(v_k - v_i + \eta)}.$$

Take first non-zero term as $\eta \to 0$: $\boxed{\textbf{QISM} \xrightarrow{\eta \to 0} \textbf{QC QISM}}$

Quasi-classical limit (QC)

Derive the expressions for $a(u)$ and $d(u)$ for $\Omega = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^{\otimes \mathcal{L}}$:

$$a(u) = e^{-\eta\gamma} \prod_{l=1}^{\mathcal{L}} \frac{\sinh(u - \varepsilon_l - \eta/2)}{\sinh(u - \varepsilon_l)}, \quad d(u) = e^{\eta\gamma} \prod_{l=1}^{\mathcal{L}} \frac{\sinh(u - \varepsilon_l + \eta/2)}{\sinh(u - \varepsilon_l)}.$$

Then the **BAE**:

$$e^{-2\eta\gamma} \prod_{l=1}^{\mathcal{L}} \frac{\sinh(v_k - \varepsilon_l - \eta/2)}{\sinh(v_k - \varepsilon_l + \eta/2)} = \prod_{i \neq k}^{N} \frac{\sinh(v_k - v_i - \eta)}{\sinh(v_k - v_i + \eta)}.$$

Take first non-zero term as $\eta \to 0$: $\boxed{\textbf{QISM} \xrightarrow{\eta \to 0} \textbf{QC QISM}}$

- **QC trig. QISM [Amico et al. 2001], [Dukelsky et al. 2001]**

$$\boxed{2\gamma + \sum_{l=1}^{\mathcal{L}} \coth(v_k - \varepsilon_l) = 2\sum_{i \neq k}^{N} \coth(v_k - v_i)} \qquad (\diamondsuit)$$

- **QC rat. QISM [Richardson 1963]**

$$\boxed{2\gamma + \sum_{l=1}^{\mathcal{L}} \frac{1}{v_k - \varepsilon_l} = \sum_{i \neq k}^{N} \frac{2}{v_k - v_i}} \qquad (\heartsuit)$$

$$\boxed{\textbf{QC trig. QISM } (\diamondsuit) \xrightarrow{\text{rational limit}} \textbf{QC rat. QISM } (\heartsuit)}$$

- Change of variables $v_i = \ln y_i,\ \varepsilon_l = \ln z_l$ in $(\diamondsuit)$ gives

$$\boxed{(N - \mathcal{L}/2 + \gamma - 1) + \sum_{l=1}^{\mathcal{L}} \frac{y_k^2}{y_k^2 - z_l^2} = \sum_{i \neq k}^{N} \frac{2y_k^2}{y_k^2 - y_i^2}} \qquad (\diamondsuit')$$

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\textbf{QC trig. BQISM} \arrow[r, dashed, shift left, "\text{attenuated limit}"] \arrow[d, dashed, shift right, "\text{rational limit}"'] & \textbf{QC trig. QISM } (\Diamond) \arrow[l, dashed, shift left, "\text{variable change}"] \arrow[d, red, "\color{red}\text{rational limit}"'] \\
\textbf{QC rat. BQISM} \arrow[u, dashed, shift right, "\text{variable change}"'] \arrow[r, dashed, "\text{attenuated limit}"] & \textbf{QC rat. QISM } (\heartsuit)
\end{tikzcd}
$$

Reflection equations [Cherednik 1984]

Start with the **trigonometric** R-matrix. In addition to the YBE we require that it satisfies the **reflection equations** for some $K^{\pm} \in \mathrm{End}(V)$, referred to as the **reflection matrices**:

$$\begin{cases} R_{12}(u-v)K_1^-(u)R_{21}(u+v)K_2^-(v) = K_2^-(v)R_{12}(u+v)K_1^-(u)R_{21}(u-v), \\ R_{12}(v-u)K_1^+(u)\mathcal{R}_{21}(u+v)K_2^+(v) = K_2^+(v)\mathcal{R}_{12}(u+v)K_1^+(u)R_{21}(v-u), \end{cases}$$

where $\mathcal{R}(u) \equiv R(-u-2\eta)$. Easy to check that

$$K^-(u) = \begin{pmatrix} \sinh(\xi^- + u) & 0 \\ 0 & \sinh(\xi^- - u) \end{pmatrix},$$

$$K^+(u) = \begin{pmatrix} \sinh(\xi^+ + u + \eta) & 0 \\ 0 & \sinh(\xi^+ - u - \eta) \end{pmatrix}$$

satisfy these equations for any $\xi^{\pm} \in \mathbb{C}$.

BQISM [Sklyanin 1988]

Define the **double row monodromy matrix** $\in \mathrm{End}\left(V_a \otimes V^{\otimes \mathcal{L}}\right)$:

$$T_a(u) := R_{a\mathcal{L}}(u-\varepsilon_{\mathcal{L}})...R_{a1}(u-\varepsilon_1)K_a^-(u)\times$$
$$\times R_{a1}^{-1}(-u-\varepsilon_1)...R_{a\mathcal{L}}^{-1}(-u-\varepsilon_{\mathcal{L}}) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}.$$

The **transfer matrix**

$$t(u) := tr_a\left(K_a^+(u)\,T_a(u)\right)$$

satisfies

$$\boxed{[t(u), t(v)] = 0 \ \ \forall u, v \in \mathbb{C}}$$

Thus, it is a generating function for the **integrals of motion**!

Algebraic Bethe Ansatz

Introduce $\tilde{a}(u) = (2u)a(u) - \eta d(u)$. Start with a reference state Ω and look for other eigenstates in the form $\Phi(v_1, ..., v_N) = C(v_1)...C(v_N)\Omega$.

Eigenvalues:

$$\Lambda(u, v_1, ..., v_N) = \tilde{a}(u)\frac{\sinh(\xi^+ + u + \eta/2)}{\sinh 2u}\prod_{k=1}^{N}\frac{\sinh(u - v_k + \eta)\sinh(u + v_k + \eta)}{\sinh(u - v_k)\sinh(u + v_k)} +$$

$$+ d(u)\frac{\sinh(2u + \eta)\sinh(\xi^+ - u + \eta/2)}{\sinh 2u}\prod_{k=1}^{N}\frac{\sinh(u - v_k - \eta)\sinh(u + v_k - \eta)}{\sinh(u - v_k)\sinh(u + v_k)},$$

BAE:

$$\boxed{\frac{\tilde{a}(v_k)}{d(v_k)\sinh(2v_k - \eta)}\frac{\sinh(\xi^+ + v_k + \eta/2)}{\sinh(\xi^+ - v_k + \eta/2)} = \prod_{i\neq k}^{N}\frac{\sinh(v_k - v_i - \eta)\sinh(v_k + v_i - \eta)}{\sinh(v_k - v_i + \eta)\sinh(v_k + v_i + \eta)}}$$

Quasi-classical limit (QC)

If we substitute $\eta = 0$ the BAE will take the following form:

$$\frac{\sinh(\xi^- + v_k)\sinh(\xi^+ + v_k)}{\sinh(\xi^- - v_k)\sinh(\xi^+ - v_k)} = 1.$$

Choose $\xi^- = \xi^-(\eta),\ \xi^+ = \xi^+(\eta)$, so that this holds as $\eta \to 0$.

Take

$$\boxed{\xi^+ = \eta\alpha, \quad \xi^- = \eta\beta}$$

- **QC trig. BQISM**

$$\boxed{\begin{aligned} &-2(\alpha+\beta+1)\coth v_k + \sum_{l=1}^{\mathcal{L}} (\coth(v_k - \varepsilon_l) + \coth(v_k + \varepsilon_l)) = \\ &= 2\sum_{i \neq k}^{N} (\coth(v_k - v_i) + \coth(v_k + v_i)) \end{aligned}} \qquad (\clubsuit)$$

- **QC rat. BQISM**

$$\boxed{-(\alpha+\beta+1)+\sum_{l=1}^{\mathcal{L}}\frac{v_k^2}{v_k^2-\varepsilon_l^2}=\sum_{i\neq k}^{N}\frac{2v_k^2}{v_k^2-v_i^2}} \qquad (\spadesuit)$$

Same expression as $(\diamondsuit')$ **QC trig. QISM** with the change of variables!

$$\boxed{\textbf{QC rat. BQISM } (\spadesuit) \Leftrightarrow \textbf{QC trig. QISM } (\diamondsuit)}$$

- **QC trig. BQISM**

Change of variables $v_k=\ln y_k,\ \varepsilon_l=\ln z_l$ in $(\clubsuit)$ gives

$$\boxed{\begin{aligned}&-(\alpha+\beta+1)\frac{y_k^2+1}{y_k^2-1}+\sum_{l=1}^{\mathcal{L}}\left(\frac{y_k^2}{y_k^2-z_l^2}+\frac{1}{y_k^2z_l^2-1}\right)=\\&=\sum_{i\neq k}^{N}\left(\frac{2y_k^2}{y_k^2-y_i^2}+\frac{2}{y_k^2y_i^2-1}\right)\end{aligned}} \qquad (\clubsuit')$$

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\textbf{QC trig. BQISM } (\clubsuit) \arrow[r, dashed, shift left, "\text{attenuated limit}"] \arrow[d, shift right, red, "\text{rational limit}"'] & \textbf{QC trig. QISM } (\diamondsuit) \arrow[l, dashed, shift left, "\text{variable change}"] \arrow[d, "\text{rational limit}"'] \\
\textbf{QC rat. BQISM } (\spadesuit) \arrow[u, dashed, shift right, "\text{variable change}"'] \arrow[r, dashed, "\text{attenuated limit}"] \arrow[ur, phantom, "{\color{red}(\spadesuit) \Leftrightarrow (\diamondsuit)}"] & \textbf{QC rat. QISM } (\heartsuit)
\end{tikzcd}
$$

Introduce an **additional parameter** ρ: $v_k \to v_k + \frac{\rho}{2}$, $\varepsilon_l \to \varepsilon_l + \frac{\rho}{2}$.

$$\boxed{\textbf{QC rat. BQISM } (\spadesuit) \xrightarrow{\rho\to\infty} \textbf{QC rat. QISM } (\heartsuit)}$$

$$-(\alpha+\beta+1) + \sum_{l=1}^{\mathcal{L}} \frac{v_k^2 + \rho v_k + \rho^2/4}{v_k^2 - \varepsilon_l^2 + \rho(v_k - \varepsilon_l)} = 2\sum_{i\neq k}^{N} \frac{v_k^2 + \rho v_k + \rho^2/4}{v_k^2 - v_i^2 + \rho(v_k - v_i)}.$$

Rescale $\alpha + \beta = \rho(\alpha' + \beta')/4$ and consider $\rho \to \infty$:

$$-(\alpha' + \beta') + \sum_{l=1}^{\mathcal{L}} \frac{1}{v_k - \varepsilon_l} = \sum_{i\neq k}^{N} \frac{2}{v_k - v_i}.$$

$\boxed{\textbf{QC trig. BQISM } (\clubsuit') \xrightarrow{\rho\to\infty} \textbf{QC trig. QISM } (\diamondsuit')}$ Set $\rho/2 = \ln \sigma$:

$$-(\alpha+\beta+1)\frac{\sigma^2 y_k^2 + 1}{\sigma^2 y_k^2 - 1} + \sum_{l=1}^{\mathcal{L}} \left(\frac{y_k^2}{y_k^2 - z_l^2} + \frac{1}{\sigma^4 y_k^2 z_l^2 - 1} \right) = \sum_{i\neq k}^{N} \left(\frac{2y_k^2}{y_k^2 - y_i^2} + \frac{2}{\sigma^4 y_k^2 y_i^2 - 1} \right)$$

and consider $\sigma \to \infty$:

$$-(\alpha+\beta+1) + \sum_{l=1}^{\mathcal{L}} \frac{y_k^2}{y_k^2 - z_l^2} = \sum_{i\neq k}^{N} \frac{2y_k^2}{y_k^2 - y_i^2}.$$

Thus, we have

$$\begin{tikzcd}
\textbf{QC trig. BQISM } (\clubsuit) \arrow[rr, "\text{attenuated limit}"] \arrow[dd, "\text{rational limit}"'] & & \textbf{QC trig. QISM } (\diamondsuit) \arrow[dd, "\text{rational limit}"'] \\
& (\spadesuit) \Leftrightarrow (\diamondsuit) & \\
\textbf{QC rat. BQISM } (\spadesuit) \arrow[rr, "\text{attenuated limit}"] & & \textbf{QC rat. QISM } (\heartsuit)
\end{tikzcd}$$

But in fact

$$\begin{tikzcd}
\textbf{QC trig. BQISM } (\clubsuit) \arrow[r, shift left, "\text{attenuated limit}"] \arrow[d, shift right, "\text{rational limit}"'] \arrow[dr, phantom, "(\spadesuit) \Leftrightarrow (\diamondsuit)"] & \textbf{QC trig. QISM } (\diamondsuit) \arrow[l, dashed, shift left, "\text{variable change}"] \arrow[d, "\text{rational limit}"'] \\
\textbf{QC rat. BQISM } (\spadesuit) \arrow[u, dashed, shift right, "\text{variable change}"'] \arrow[r, "\text{attenuated limit}"] & \textbf{QC rat. QISM } (\heartsuit)
\end{tikzcd}$$

Variable change

Substitute $\boxed{v_k \to y_k - y_k^{-1},\ \varepsilon_l \to z_l - z_l^{-1}}$ into (♠):

$$-(\alpha+\beta+1)\frac{y_k + y_k^{-1}}{y_k - y_k^{-1}} + \sum_{l=1}^{\mathcal{L}} \frac{y_k^2 - y_k^{-2}}{y_k^2 + y_k^{-2} - z_l^2 - z_l^{-2}} = 2\sum_{i\neq k}^{N} \frac{y_k^2 - y_k^{-2}}{y_k^2 + y_k^{-2} - y_i^2 - y_i^{-2}}.$$

Using

$$\frac{y_k^2 - y_k^{-2}}{y_k^2 + y_k^{-2} - z_l^2 - z_l^{-2}} = \frac{y_k^2}{y_k^2 - z_l^2} + \frac{1}{y_k^2 z_l^2 - 1}$$

we obtain (♣′):

$$-(\alpha+\beta+1)\frac{y_k^2 + 1}{y_k^2 - 1} + \sum_{l=1}^{\mathcal{L}} \left(\frac{y_k^2}{y_k^2 - z_l^2} + \frac{1}{y_k^2 z_l^2 - 1} \right) = \sum_{i\neq k}^{N} \left(\frac{2y_k^2}{y_k^2 - y_i^2} + \frac{2}{y_k^2 y_i^2 - 1} \right).$$

$$\boxed{\textbf{QC rat. BQISM (♠)} \xrightarrow{\text{variable change}} \textbf{QC trig. BQISM (♣)}}$$

Thank you for your attention!

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\textbf{QC trig. BQISM } (\clubsuit) \arrow[r, shift left, "\text{attenuated limit}"] \arrow[d, shift right, "\text{rational limit}"'] \arrow[dr, phantom, "(\spadesuit) \Leftrightarrow (\diamondsuit)"] & \textbf{QC trig. QISM } (\diamondsuit) \arrow[l, shift left, red, "\text{variable change}"] \arrow[d, "\text{rational limit}"'] \\
\textbf{QC rat. BQISM } (\spadesuit) \arrow[u, shift right, red, "\text{variable change}"'] \arrow[r, "\text{attenuated limit}"] & \textbf{QC rat. QISM } (\heartsuit)
\end{tikzcd}
$$