Geometric structure of percolation clusters

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Outline

Fractal structure of percolation clusters

Our results

Conclusion
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Percolation

Bond percolation on $L \times L$ square lattice

Edges are independently occupied with probability $p$

Phase transition $p_c = 1/2$

$P_\infty \sim (p - p_c)^{\beta}$ for $p \to p_c^+$

Correlation length $\xi \sim |p - p_c|^{-\nu}$

$\beta = 5/36$, $\nu = 4/3$
Percolation

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\[\text{Fractal structure of percolation clusters} \quad \text{Our results} \quad \text{Conclusion} \]
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- \( p_c = 1/2 \)
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- Correlation length
  \( \xi \sim |p - p_c|^{-\nu} \)
- \( \beta = 5/36, \nu = 4/3 \)
Fractal structure

- Mean size of the largest cluster \( \sim L^{d_F}, \quad d_F = \frac{91}{48} \)
- Backbone
  - Mean size of backbone \( \sim L^{d_B}, \quad d_B = 1.64336(10) \)
- Red bond
  - Mean number of red bonds \( \sim L^{d_R}, \quad d_R = \frac{3}{4} \)
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Fractal structure of percolation clusters

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Bonds partition

Partition all bonds into branches, junctions and non-bridges

- Branches: bridges and deletion of which produces trees (green)
- Junctions: bridges but not branches (red)
- Non-bridges: not bridges (black)

- Branch density $\rho_g$
- Junction density $\rho_j$
- Non-bridge density $\rho_n$
- Leaf-free clusters
- Bridge-free clusters
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- Bond configuration → loop configuration
- Loops drawn on medial graph
- Mean length of the largest loop $\sim L^{d_H}$, with $d_H = 7/4$
- Accessible external perimeter $\sim L^{d_E}$, with $d_E = 4/3$
- Bonds bounded by the same loop, density $\rho_1$
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For trivial topology

- $\rho_1 = \rho_g + \rho_j$
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- pseudo-bridge
- $\rho_1 > \rho_g + \rho_j$
- $\rho_2 < \rho_n$
Findings

- Denote the original graph and the dual graph as $G$ and $G^*$
- $m = |E(G)| = |E(G^*)|
- Let $A \subset E(G)$ and define $A^* \subset E(G^*)$ via $e^* \in A^*$ iff $e \notin A$
- Let $\ell_1(e)$ be the event that $e$ is bounded by the same loop
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Findings

**Lemma:** On the torus $\rho_1 = \rho_2 = 1/4$ at $p_c = 1/2$

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Let $|A| = a$, $e \in A$, and $B^* = A^* \cup e^*$
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One-to-one correspondence between $(A, e)$ and $(B^*, e^*)$ such that

$$1_{\ell_1(e)}(A) = 1 \iff 1_{\ell_2(e^*)}(B^*) = 1.$$
Findings

Lemma: On the torus $\rho_1 = \rho_2 = \frac{1}{4}$ at $p_c = \frac{1}{2}$

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Findings

- This gives

\[ \sum A \subseteq E \sum_{|A|=a} 1_{\ell_1}(e)(A) = \sum B^* \subseteq E^* \sum_{|B^*|=m+1-a} 1_{\ell_2}(e^*)(B^*) \]

- Summing over all \(a\) and dividing by \(\frac{1}{m2^m}\) implies \(\rho_1 = \rho_2\)

- Since \(\rho_1 + \rho_2 = 1/2\), we have \(\rho_1 = \rho_2 = 1/4\)
Findings

- This gives

\[
\sum_{A \subseteq E} \sum_{e \in A} 1_{\ell_1(e)}(A) = \sum_{B^* \subseteq E^*} \sum_{e^* \in B^*} 1_{\ell_2(e^*)}(B^*)
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- Summing over all \( a \) and dividing by \( \frac{1}{m^{2m}} \) implies \( \rho_1 = \rho_2 \)

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Findings

- On the torus, we expect:
  \[ \rho_n \rightarrow \rho_2 = 1/4, \text{ as } L \rightarrow \infty. \]

- For finite $L$, $\rho_n = \rho_{n,0} + b_1 L^{y_1}$.

- $\rho_{n,0} = 0.250000^{+1}_{-2}$
  
- $y_1 = -1.250^{+1}_{-0}$

- $y_1$ consistent with
  
  \[ d_R - 2 = -5/4 \]

- Number of pseudobridges
  
  \[ L^2(\rho_n - \rho_2) \sim L^{d_R} \]
Findings

Non-bridge density $\rho_n$

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- $y_1$ consistent with $d_R - 2 = -5/4$
- Number of pseudobridges $L^2(\rho_n - \rho_2) \sim L^{d_R}$
Non-bridge density $\rho_n$

- On the torus, we expect:
  $$\rho_n \to \rho_2 = 1/4, \text{ as } L \to \infty.$$  
- For finite $L$, $\rho_n = \rho_{n,0} + b_1 L^{y_1}$.

$\rho_n$ vs. $L^{-5/4}$

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Findings

▶ On the torus, we expect:

\[ \rho_g + \rho_j \to \rho_1 = \frac{1}{4}, \text{ as } L \to \infty. \]

▶ For finite \( L \), \( \rho_g(\rho_j) = \rho_{g,0}(\rho_{j,0}) - b_1 L^{y_1}. \)

▶ \( \rho_{j,0} = 0.035\,949\,79(8) \)

▶ \( \rho_{g,0} = 0.214\,050\,18(5) \)

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Findings

Branch density $\rho_g$ and junction density $\rho_j$

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Findings

- For leaf-free clusters
  - fractal dimension for clusters is $d_F = 91/48$
  - fractal dimension for loops is $d_H = 7/4$

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  - fractal dimension for clusters is $d_B = 1.64336(10)$
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Outline

Fractal structure of percolation clusters

Our results

Conclusion
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- Partition bonds into branches, junctions and non-bridges
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What happens for the general Fortuin-Kasteleyn random-cluster model?
Many thanks for your attention!
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