Newtonian limits and the evolution of inhomogeneous universes

Calum Robertson

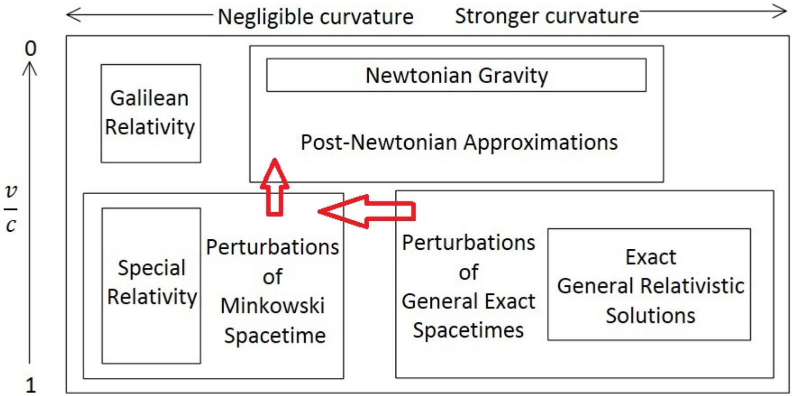
School of Mathematical Sciences, Monash University

Australia New Zealand Mathematics Convention, 2014

Outline

1. Gravity on different scales
   - General Relativity & Newtonian Gravity
   - Relevance to cosmology
2. Constructing inhomogeneous solutions
   - Background and foreground
   - Newtonian limits

Schematic of regimes



Gravitational field equations

Fundamental variable in GR is the metric **g**, components g_{ij}.

Covariant derivative (Levi-Civita) operator ∇ is obtained from **g**.

GR's equations ("EFEs")

$$G^{ij} = \frac{8\pi G}{c^4} T^{ij} - \Lambda g^{ij} \quad \text{and} \quad \nabla_i T^{ij} = 0,$$

where $\nabla_i G^{ij} = 0$ and $\nabla_i g^{ij} = 0$ hold automatically.
Given T^{ij}, solutions determined up to choice of coordinates/basis.

Newtonian gravity (for comparison)

$$\triangle \Phi = (4\pi G) \rho,$$

where $\triangle$ is the (flat) spatial Laplacian operator.
Linearised GR against a flat background recovers this.

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Constraints and initial data

Gravitational constraints: partial redundancy of Bianchi identity and EFEs

- Close look at EFEs: $G^{0j} = T^{0j}$ equations do not evolve second order initial data - they constrain it instead.
- Constraints automatically propagate!

Gauge constraints

- Determine which member (or subclass) of g's isometry class we are looking at.
- Make evolution equations easier to use.

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Inhomogeneous source matter

- Newtonian gravity still easy to use without symmetry
  $\rightarrow$ use this to build inhomogeneous model.
- Inhomogeneous structure formation
  - Millennium (2005-2010, dark matter only)
  - Illustris (current, baryonic matter included)
- Incorporate "Post-Newtonian" corrections to approximate relativity
- Obtain these by limiting GR into Newtonian physics (base: FLRW exact solution)

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Cosmological fluid

- Stress-energy tensor for perfect Euler fluid, equation of state $p = f(\rho)$ controls speed of sound:

$$T^{ij} = (\rho + f(\rho))\, v^i v^j + (f(\rho) - \Lambda)\, g^{ij}$$

- Variables of Einstein-Euler system:

$$\{\mathbf{g}, \mathbf{v}, \rho\}$$

- Newtonian limit exists $\iff$ family of solns limit to Poisson-Euler equations as $\varepsilon \searrow 0$, where $\varepsilon \sim \frac{v_T}{c}$:

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- FLRW has "cosmic time" $\rightarrow$ Newtonian universal time.

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Conformal transformations & perturbations

- "Background" **h** and its connection $\bar{D}$ used as reference point for:

$$g^{ij} = \Omega^2 \bar{g}^{ij} = \Omega^2 \left(h^{ij} + \varepsilon u^{ij}\right)$$

- Main grav variable: "foreground" **u**. Auxiliary variable Ω.

Gravity

- Generalised harmonic coordinate conditions: $\bar{g}^{pq}\bar{\nabla}_p\bar{\nabla}_q x^k = \beta^k$
- Effectively, β constrains relationship between full connection $\bar{\nabla}$, and $\bar{D}$.
- Dynamics: $G^{ij}_{\mathbf{u}} = \bar{G}^{ij} - G^{ij}_{\mathbf{h}}$

Matter

- $\bar{T}^{ij}$ now depends on $\{\bar{\mathbf{g}}, \bar{\mathbf{v}}, \bar{\rho}, \Omega, \bar{\nabla}\Omega, \bar{\nabla}^2\Omega\}$
- $\bar{p} = \varepsilon^2 f(\bar{\rho})$ keeps sound non-relativistic
- $\bar{\nabla}^2\Omega$ conformal evolution, free to specify

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Free and constrained initial data

- Specify $\mathbf{h}$ including coordinate representation $\rightarrow$ gauge constraints transfer coordinate choice to $\bar{\mathbf{g}}$
- Initial data: $\left\{\mathbf{u}(0), D_0\mathbf{u}(0), \bar{\mathbf{v}}(0), \bar{\rho}(0), \Omega(0), \bar{\nabla}_0\Omega(0)\right\}$.
- Partially constrained by $\bar{G}^{0j} = \bar{T}^{0j}$, $\beta^k = g^{ab}\left(\Gamma(\bar{\mathbf{g}})^k_{ab} - \Gamma(\mathbf{h})^k_{ab}\right)$.
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- Not so much the $\varepsilon \searrow 0$ limit as it is the $\varepsilon\mathbf{u} \searrow 0$ limit.
- Geometry should approach background geometry
  - Light cone structure approaches that of background
- Classical linearised gravity recoverable from $\bar{G}^{ij} = \bar{T}^{ij}$, when:
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Here's one we prepared earlier (flat background)

- Background coords: $x^0 = \bar{x}^0, \quad x^I = \frac{1}{\varepsilon}\bar{x}^I$
- Spatial derivatives pick up negative power of ε
- Variables have ε-expansions: iterative solution
- Before applying scaling (Ω incorporates trace reversal):

$$\varepsilon\left(\bar{g}^{kl}\bar{\partial}_k\bar{\partial}_l\bar{u}^{ij} + \bar{g}^{ij}\bar{\partial}_k\bar{\beta}^k - 2\bar{g}^{k(i}\bar{\partial}_k\bar{\beta}^{j)}\right) + \varepsilon^2 Q(\partial\mathbf{u})$$
$$= \tau^{ij} - \frac{2}{\Omega}\left(\bar{g}^{ik}\bar{g}^{jl} - \bar{g}^{ij}\bar{g}^{kl}\right)\bar{\nabla}_k\bar{\nabla}_l\Omega - \frac{3}{\Omega^2}\bar{g}^{ij}\bar{g}^{kl}\left(\bar{\nabla}_k\Omega\right)\left(\bar{\nabla}_l\Omega\right)$$

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(Simplified) summary: singular $\varepsilon \searrow 0$ limit

1. Structure of gravitational dynamics generically given as per prev calculations
2. Apply Newtonian scaling to $\bar{T}^{ij}$ & cancel out background dynamics ($G_{\mathbf{h}}^{ij}$)
3. Expand $\nabla_i \bar{T}^{ij} = 0$, expand matter variables as appropriate (case by case!)
4. Use Ω-freedom to control remaining singular behaviour
5. Use β-freedom to modify evolution equations as needed
6. Define initial data & enforce constraints
7. Obtain conditions upon free initial data for limit to exist
8. ...compare simulations of relativistic & PN solutions

References & Acknowledgments

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